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On some mixed types of continuity on generalized neighborhood systems

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Abstract

The aim of this paper is to introduce mixed almost $(\psi, \psi_1\psi_2)$-continuity and mixed weakly $(\psi, \psi_1\psi_2)$-continuity between a generalized neighborhood system ψ and two generalized neighborhood systems ψ_1, ψ_2. We investigate their properties and several examples are provided to illustrate the behavior of between these new types of mixed continuity and some other mixed continuities on generalized topological spaces.

Keywords: almost (μ, g_1g_2)-continuous, weakly $(\psi, \psi_1\psi_2)$-continuous, almost $(\psi, \psi_1\psi_2)$-continuous.
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1. Introduction

The notion of continuity is one of the most important concepts in mathematics. In the generalized topology, recently have seen the introduction of many types of generalized continuity and mixed continuity on generalized topological spaces.

Á. Császár [1] introduced the generalized topological spaces and generalized neighborhood systems. He also obtained (ψ, ψ')-continuity and (g, g')-continuity on generalized neighborhood systems and generalized topological spaces, respectively. In recent years, the notions of mixed weakly (μ, g_1g_2)-continuity, mixed $\theta(\mu, g_1g_2)$-continuity and mixed almost (μ, g_1g_2)-continuity are defined by Min [5, 6, 7].

The purpose of the present paper is the introduce and study the notions of mixed almost $(\psi, \psi_1\psi_2)$-continuity and mixed weakly $(\psi, \psi_1\psi_2)$-continuity on generalized neighborhood systems. We discuss some properties and characterizations of these mixed continuities. In particular, we investigate the relationships among (ψ, ψ_1)-continuity, (μ, g_1)-continuity, weakly (μ, g_1g_2)-continuity, mixed almost (μ, g_1g_2)-continuity and these mixed continuities on generalized neighborhood systems.

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2. Preliminaries

We briefly touch upon some definitions, theorems and results, which are needed in the sequel. The following are needed in our study and can be found in the paper referred to.

2.1. Generalized Topology

Definition 2.1. [1] Let $X \neq \emptyset$ and $g \subseteq X$. Then g is called a generalized topology (briefly, GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i\in I} G_i \in g$. The pair (X, g) is called a generalized topological space (briefly, GTS) on X. The elements of g are called g-open sets and their complements are called g-closed sets.

Set $gO(X)$ and $gO(x)$ are defined as $gO(X) = \{U \subseteq X : U \in g\}$, $gO(x) = \{U \in g : x \in g\}$, respectively.

Definition 2.2. [1] The generalized-interior of a subset S of X, denoted by $i_g(S)$, is the union of generalized open sets included in S, and the closure of S, denoted by $c_g(S)$, is the intersection of generalized closed sets including S.

Theorem 2.3. [1] Let (X, g) be a generalized topological space. Then the following hold:

(1) $c_g(A) = X - i_g(X - A)$.
(2) $i_g(A) = X - c_g(X - A)$.

Definition 2.4. [1] Let g_1 and g_2 be two GT's on nonempty sets on X and Y, respectively. $f : X \to Y$ is said to be (g_1, g_2)-continuous if $f^{-1}(G) \in g_1$ for all $G \in g_2$.

Definition 2.5. Let g_1 and g_2 be two GT's on a nonempty set X. Then A is said to be:

(1) (g_1, g_2)-semiopen [2] if $A \subseteq c_{g_2}(i_{g_1}(A))$.
(2) (g_1, g_2)-preopen [2] if $A \subseteq i_{g_1}(c_{g_2}(A))$.
(3) (g_1, g_2)-β'-open [2] if $A \subseteq c_{g_2}(i_{g_1}(c_{g_2}(A)))$.
(4) $r(g_1, g_2)$-open (or (g_1, g_2)-regular open) (resp., $r(g_1, g_2)$-closed) (or (g_1, g_2)-reguler closed) [3] if $A = i_{g_1}(c_{g_2}(A))$ (resp., $A = c_{g_1}(i_{g_2}(A))$).

The complement of (g_1, g_2)-semiopen (resp., (g_1, g_2)-preopen, (g_1, g_2)-β'-open) is called (g_1, g_2)-semi closed (resp., (g_1, g_2)-preclosed, (g_1, g_2)-β'-closed).

Definition 2.6. [6] Let g_1 and g_2 be two GT's on a nonempty set X. Then A is said to be:

(1) $(g_1, g_2)'$-semiopen if $A \subseteq c_{g_1}(i_{g_2}(A))$.
(2) $(g_1, g_2)'$-preopen if $A \subseteq i_{g_2}(c_{g_1}(A))$.
(3) $(g_1, g_2)'$-β'-open if $A \subseteq c_{g_1}(i_{g_2}(c_{g_1}(A)))$.

The complement of $(g_1, g_2)'$-semiopen (resp., $(g_1, g_2)'$-preopen, $(g_1, g_2)'$-β'-open) is called $(g_1, g_2)'$-semiclosed (resp., $(g_1, g_2)'$-preclosed, $(g_1, g_2)'$-β'-closed).

Definition 2.7. [6] Let μ be a GT on a nonempty set X and g_1 , g_2 be two GT's on a nonempty set Y. Then a function $f : X \to Y$ is said to be mixed weakly (μ, g_1g_2)-continuous at $x \in X$ if for each g_1-open set V containing $f(x)$, there exists a μ-open U containing x such that $f(U) \subseteq c_{g_2}(V)$. Then f is said to be mixed weakly (μ, g_1g_2)-continuous (briefly; weakly (μ, g_1g_2)-continuous) if it is mixed weakly (μ, g_1g_2)-continuous at every point of X.

Definition 2.8. Let g_1 and g_2 be two GT's on a nonempty set X. Let $\theta(g_1, g_2)$ [3] be composed of the sets $A \subseteq X$ such that $x \in A$ implies the existence of a set $M \in g_1$ satisfying $x \in M \subseteq c_{g_2}(M) \subseteq A$. Then $\theta(g_1, g_2)$ is a GT contained in g_1 on X [3]. The element of $\theta(g_1, g_2)$ is said to be $\theta(g_1, g_2)$-open set and the complement of $\theta(g_1, g_2)$-open set is said to be $\theta(g_1, g_2)$-closed set.

$$c_{\theta(g_1,g_2)}(A) = \bigcap\{F \subseteq X : A \subseteq F \quad \text{for} \quad \theta(g_1, g_2)\text{-closed set } F \text{ in } X\} \quad [6],$$

$$i_{\theta(g_1,g_2)}(A) = \bigcup\{V \subseteq X : V \subseteq A \quad \text{for } \theta(g_1, g_2)\text{-open set } V \text{ in } X\} \quad [6],$$

$$\gamma_{\theta(g_1,g_2)}(A) = \{x \in X : c_{g_2}(M) \cap A \neq \emptyset \quad \text{for every } M \in g_1 \text{ containing } x\} \quad [3].$$

Theorem 2.9. [3] Let g_1 and g_2 be two GT's on a nonempty set X and $A \subseteq X$. Then the following hold:

(1) $A \subseteq \gamma_{\theta(g_1,g_2)}(A) \subseteq c_{\theta(g_1,g_2)}(A)$.
(2) A is $\theta(g_1, g_2)$-closed iff $A = \gamma_{\theta(g_1,g_2)}(A)$.

Lemma 2.10. [6] Let g_1 and g_2 be two GT's on a nonempty set X and $A \subseteq X$. Then the following hold:

(1) $x \in i_{\theta(g_1,g_2)}(A)$ if and only if there exists a g_1-open set M containing x such that $x \in M \subseteq c_{g_2}(M) \subseteq A$.
(2) If A is g_2-open in X, then $\gamma_{\theta(g_1,g_2)}(A) = c_{g_1}(A)$.

Definition 2.11. [6] Let g_1 and g_2 be two GT's on a nonempty set X. X is said to be (g_1, g_2)-regular if for $x \in X$ and any g_1-closed set F with $x \notin F$, there exist $U \in g_1$, $V \in g_2$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.

Theorem 2.12. [6] Let g_1 and g_2 be two GT's on a nonempty set X. If X is (g_1, g_2)-regular, then every g_1-open set is $\theta(g_1, g_2)$-open.

Definition 2.13. [2] Let g_1, g_2 be two GT's on a nonempty set X. Then g_1 is g_2-EDC iff $G \in g_1$ implies that $c_{g_2}(G) \in g_1$.

2.2. Generalized Neighborhood Systems

Definition 2.14. [1] Let $\psi : X \to expexpX$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a generalized neighborhood of $x \in X$ and ψ is called a generalized neighborhood system (briefly, GNS) on X.

Denote the set of all generalized neighborhood systems on X by $\Psi(X)$.

Definition 2.15. [1] If $\psi \in \Psi(X)$ and $A \subseteq X$, the interior of A on ψ, denoted by $\iota_\psi(A)$, is the set of all $x \in X$ such that there is $V \in \psi(x)$ implies $V \subseteq A$ and the closure of A on ψ, denoted by $\gamma_\psi(A)$, is set of all $x \in X$ such that all $V \in \psi(x)$ implies $V \cap A \neq \emptyset$.

Lemma 2.16. [1] Let $\psi \in \Psi(X)$ and $G \in g_\psi$ iff $G \subseteq X$ satisfy: if $x \in G$ then there is $V \in \psi(x)$ such that $V \subseteq G$. For $\psi \in \Psi(X)$, i_ψ and c_ψ are defined as $i_\psi = i_{g_\psi}$, $c_\psi = c_{g_\psi}$, respectively.

Theorem 2.17. [1] Let $\psi \in \Psi(X)$ and $A \subseteq X$. Then the following hold:

(1) $\gamma_\psi(A) = X - \iota_\psi(X - A)$ and $\iota_\psi(A) = X - \gamma_\psi(X - A)$.
(2) $i_\psi(A) \subseteq \iota_\psi(A)$ and $\gamma_\psi(A) \subseteq c_\psi(A)$.

Lemma 2.18. [1] If $\psi \in \Psi_g(X)$ for the GT $g = g_\psi$ on X, $\iota_\psi = i_\psi$ and $\gamma_\psi = c_\psi$.

Lemma 2.19. [1] If g is a GT on X, there is $\psi \in \Psi(X)$ satisfying $g = g_\psi$. ψ satisfy $V \in g$ for $V \in \psi(x)$, $x \in X$.

Lemma 2.20. [1] If g is a GT on X and $\psi = \psi_g$ then $g_\psi = g$.

Definition 2.21. [1] Let $\psi_1 \in \Psi(X)$ and $\psi_2 \in \Psi(Y)$. $f : X \to Y$ is said to be (ψ_1, ψ_2)-continuous if for $x \in X$ and $V \in \psi_2(f(x))$, there is $U \in \psi_1(x)$ such that $f(U) \subseteq V$.

3. Mixed almost (μ, g_1g_2)-continuous functions on generalized topology

In this section, we investigate some characterizations of the mixed almost continuous functions given by Min [7].

Definition 3.1. [7] Let μ be a GT on a nonempty set X, and g_1, g_2 be two GT's on a nonempty set Y. Then a function $f : X \to Y$ is said to be mixed almost (μ, g_1g_2)-continuous at $x \in X$ if for each g_1-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq i_{g_1}(c_{g_2}(V))$. Then f is said to be mixed almost (μ, g_1g_2)-continuous if it is mixed almost (μ, g_1g_2)-continuous at every point of X.

Let (X, g_1) and (Y, g_2) be generalized topological spaces. Then a function $f : X \to Y$ is said to be almost (g_1, g_2)- continuous at $x \in X$ if for each g_2-open set V containing $f(x)$, there exists a g_1-open set U containing x such that $f(U) \subseteq i_{g_2}(c_{g_2}(V))$ [4].

Theorem 3.2. [7] Let $f : X \to Y$ be a function, μ be a GT on a nonempty set X and let g_1, g_2 be two GT's on a nonempty set Y. Then the following are equivalent:

(1) f is mixed almost (μ, g_1g_2)-continuous.
(2) $f^{-1}(V) \subseteq i_\mu(f^{-1}(i_{g_1}(c_{g_2}(V))))$ for every g_1-open subset V in Y.
(3) $c_\mu(f^{-1}(c_{g_1}(i_{g_2}(F)))) \subseteq f^{-1}(F)$ for every g_1-closed subset F in Y.
(4) For every $r(g_1, g_2)$-closed subset F in Y, $f^{-1}(F)$ is μ-closed.
(5) For every $r(g_1, g_2)$-open subset V in Y, $f^{-1}(V)$ is μ-open.

Theorem 3.3. Let $f : X \to Y$ be a function, μ be a GT on a nonempty set X and g_1, g_2 be two GT's on a nonempty set Y. Then the following are equivalent:

(1) f is mixed almost (μ, g_1g_2)-continuous at $x \in X$.
(2) For any $r(g_1, g_2)$-open subset V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let V be any $r(g_1, g_2)$-open subset of Y containing $f(x)$. By hypothesis, there exists a μ-open subset U of X containing x such that $f(U) \subseteq i_{g_1}(c_{g_2}(V))$. Since V is $r(g_1, g_2)$-open, we have $f(U) \subseteq V$.

(2) $\Rightarrow$ (1): Let V be any g_1-open subset of Y containing $f(x)$. Since $i_{g_1}(c_{g_2}(V))$ is $r(g_1, g_2)$-open, there exists a μ-open set U containing x such that $f(U) \subseteq i_{g_1}(c_{g_2}(V))$. Hence f is mixed almost (μ, g_1g_2)- continuous.

Theorem 3.4. Let $f : X \to Y$ be a function, μ be a GT on a nonempty set X and g_1, g_2 be two GT's on a nonempty set Y. Then the following are equivalent:

(1) f is mixed almost (μ, g_1g_2)-continuous.
(2) $c_\mu(f^{-1}(G)) \subseteq f^{-1}(c_{g_1}(G))$ for every $(g_1, g_2)'$-β'-open set G of Y.
(3) $c_\mu(f^{-1}(G)) \subseteq f^{-1}(c_{g_1}(G))$ for every $(g_1, g_2)'$-semiopen set G of Y.

Proof. (1) $\Rightarrow$ (2): Let G be any $(g_1, g_2)'$-β'-open set. Since $c_{g_1}(G)$ is $r(g_1, g_2)$-closed, by Theorem 3.2. (4), we have $f^{-1}(c_{g_1}(G)) = c_\mu(f^{-1}(c_{g_1}(G)))$. Hence

$$c_\mu(f^{-1}(G)) \subseteq c_\mu(f^{-1}(c_{g_1}(G))) = f^{-1}(c_{g_1}(G)).$$

(2) $\Rightarrow$ (3): It is obvious since every $(g_1, g_2)'$-semiopen set is $(g_1, g_2)'$-β'-open.

(3) $\Rightarrow$ (1): Let F be any $r(g_1, g_2)$-closed set. Then since F is $(g_1, g_2)'$-semiopen, we have $c_\mu(f^{-1}(F)) \subseteq f^{-1}(c_{g_1}(F)) = f^{-1}(F)$. By Theorem 3.2 (4), f is mixed almost (μ, g_1g_2)-continuous.

Theorem 3.5. Let $f : X \to Y$ be a function, μ be a GT on a nonempty set X and g_1, g_2 be two GT's on a nonempty set Y. Then the following are equivalent:

(1) f is mixed almost (μ, g_1g_2)-continuous.
(2) $f(c_\mu(A)) \subseteq \gamma_{\theta(g_1,g_2)}(f(A))$ for every subset A of X.
(3) $c_\mu(f^{-1}(B)) \subseteq f^{-1}(\gamma_{\theta(g_1,g_2)}(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): For $A \subseteq X$, let $x \in c_\mu(A)$ and V any g_1-open set containing $f(x)$. By hypothesis, there exists a μ-open set U containing x such that $f(U) \subseteq i_{g_1}(c_{g_2}(V))$. This implies $f(U) \subseteq i_{g_1}(c_{g_2}(V) \subseteq c_{g_2}(V)$. Since $x \in c_\mu(A)$, $A \cap U \neq \emptyset$ for the μ-open set U containing x. So $\emptyset \neq f(U) \cap f(A) \subseteq c_{g_2}(V) \cap f(A)$. This implies $f(x) \in \gamma_{\theta(g_1,g_2)}(f(A))$. Hence we have $f(c_\mu(A)) \subseteq \gamma_{\theta(g_1,g_2)}(A)$.

(2) ⇒ (3): It is obvious.

(3) ⇒ (1): Let V be any g_2-open set. Then from Lemma 2.10 (2), we know that $\gamma_{\theta(g_1,g_2)}(V) = c_{g_1}(V)$. By hypothesis,

$$c_\mu(f^{-1}(V)) \subseteq f^{-1}(\gamma_{\theta(g_1,g_2)}(V)) = f^{-1}(c_{g_1}(V)).$$

Since every g_2-open set is $(g_1, g_2)'$-semiopen set, by Theorem 3.4 (3) f is mixed almost (μ, g_1g_2)-continuous.

Theorem 3.6. Let $f : X \to Y$ be a function, μ be a GT on a nonempty set X and g_1, g_2 be two GT's on a nonempty set Y. If Y is (g_1, g_2)-regular, then the following are equivalent:

(1) f is mixed almost (μ, g_1g_2)-continuous.
(2) $f^{-1}(B)$ is μ-closed for every $\theta(g_1, g_2)$-closed subset B of Y.
(3) $f^{-1}(V)$ is μ-open for every $\theta(g_1, g_2)$-open subset V of Y.

Proof. (1) ⇒ (2): Let B be any $\theta(g_1, g_2)$-closed subset of Y. From Theorem 2.9 (2) and Theorem 3.5, we have

$$c_\mu(f^{-1}(B)) \subseteq f^{-1}(\gamma_{\theta(g_1,g_2)}(B)) = f^{-1}(B).$$

Hence $f^{-1}(B)$ is μ-closed.

(2) ⇒ (3): It is obvious.

(3) ⇒ (1): Let V be any g_1-open subset of Y. By Y is (g_1, g_2)-regular and Theorem 2.12, V is $\theta(g_1, g_2)$-open. By hypothesis,

$$f^{-1}(V) = i_\mu(f^{-1}(V)) \subseteq i_\mu(f^{-1}(i_{g_1}(c_{g_2}(V)))).$$

Thus from Theorem 3.2, f is mixed almost (μ, g_1g_2)-continuous.

Theorem 3.7. Let μ_1 be a GT on a nonempty set X, μ_2 be a GT on a nonempty set Y and g_1, g_2 be two GT's on a nonempty set Z. If $f : X \to Y$ is (μ_1, μ_2)-continuous and $g : Y \to Z$ is mixed almost (μ_2, g_1g_2)-continuous, then the composition $gof : X \to Z$ is mixed almost (μ_1, g_1g_2)-continuous.

Proof. Let $x \in X$ and G be any g_1-open subset of Z containing $g(f(x))$. Since g is mixed almost (μ_2, g_1g_2)-continuous, there exists a μ_2-open subset U of X containing $f(x)$ such that $g(U) \subseteq i_{g_1}(c_{g_2}(G))$. Since f is (μ_1, μ_2)-continuous, $f^{-1}(U) \in \mu_1$. For $V = f^{-1}(U)$, $(gof)(V) = g(f(f^{-1}(U))) \subseteq g(U) \subseteq i_{g_1}(c_{g_2}(G))$. Thus, gof is mixed almost (μ_1, g_1g_2)-continuous.

We recall the notion of the graph function. Let $f : X \to Y$ be a function. The graph function $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$.

Theorem 3.8. Let μ be a GT on a nonempty set X and g_1, g_2 be two GT's on a nonempty set Y. If the graph function $g : X \to X \times Y$ is mixed almost (μ, g_1g_2)-continuous, then $f : X \to Y$ is mixed almost (μ, g_1g_2)-continuous.

Proof. Let $x \in X$ and G be any g_1-open subset of Y containing $f(x)$. Then for $x \in V \in \mu$, $V \times G$ is an open set containing $g(x)$. Since g is mixed almost (μ, g_1g_2)-continuous, there exists a μ-open subset U of X containing x such that $g(U) \subseteq i_{g_1}(c_{g_2}(V \times G))$. For $V \in \mu$, $g(U) \subseteq i_{g_1}(c_{g_2}(V \times G)) = i_{g_1}(c_{g_2}(V)) \times i_{g_1}(c_{g_2}(G))$ and $f(U) \subseteq i_{g_1}(c_{g_2}(G))$. Consequently, f is mixed almost (μ, g_1g_2)-continuous.

4. Mixed almost and weakly $(\psi, \psi_1\psi_2)$-continuous functions on generalized neighborhood systems

In this section, we introduce the notions of mixed almost $(\psi, \psi_1\psi_2)$-continuity and mixed weakly $(\psi, \psi_1\psi_2)$-continuity on generalized neighborhood systems, and establish some characterizations for such functions.

Definition 4.1. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\ \psi_2 \in \Psi(Y)$. Then f is said to be mixed almost $(\psi, \psi_1\psi_2)$-continuous at $x \in X$ if for $x \in X$ and $V \in \psi_1(f(x))$, there exists $U \in \psi(x)$ such that $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. Then f is said to be mixed almost $(\psi, \psi_1\psi_2)$-continuous (briefly, almost $(\psi, \psi_1\psi_2)$-continuous) if it is mixed almost $(\psi, \psi_1\psi_2)$-continuous at every point of X.

Remark 4.2. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\ \psi_2 \in \Psi(Y)$. If f is (ψ, ψ_1)- continuous, then it is almost $(\psi, \psi_1\psi_2)$-continuous. But the converse is not always true as shown in the next example.

Example 4.3. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Consider generalized neighborhood systems $\psi \in \Psi(X)$ and $\psi_1, \psi_2 \in \Psi(Y)$ defined as follows:

$$\psi(1) = \{\{1\}, \{1, 2\}\},\ \psi(2) = \{\{1, 2\}\},\ \psi(3) = \{\{1, 3\}\}.$$

$$\psi_1(a) = \{\{a, b\}\},\ \psi_1(b) = \{\{b\}\},\ \psi_1(c) = \{\{a, c\}\}.$$

$$\psi_2(a) = \{Y\},\ \psi_2(b) = \{Y\},\ \psi_2(c) = \{Y\}.$$

A function $f : X \to Y$ defined as $f(1) = a$, $f(2) = b$, $f(3) = c$. Then f is almost $(\psi, \psi_1\psi_2)$-continuous. But, f is not (ψ, ψ_1)-continuous at point 2, and so f is not (ψ, ψ_1)-continuous.

Definition 4.4. Let $\psi_1,\ \psi_2 \in \Psi(X)$ and $x \in X$. A generalized neighborhood system ψ_1 is said to be ψ_2-closed if $\gamma_{\psi_2}(V) = V$ for all $V \in \psi_1(x)$.

Theorem 4.5. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1, \psi_2 \in \Psi(Y)$. If ψ_1 is ψ_2- closed, then the following are equivalent:

(1) f is (ψ, ψ_1)-continuous.
(2) f is almost $(\psi, \psi_1\psi_2)$-continuous.

Proof. (1) ⇒ (2): Obvious.

(2) ⇒ (1): Let $V \in \psi_1(x)$ for $x \in X$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. Since ψ_1 is ψ_2-closed, $\gamma_{\psi_2}(V) = V$. Since $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V)) \subseteq \gamma_{\psi_2}(V) = V$ for $U \in \psi(x)$, f is (ψ, ψ_1)-continuous.

Theorem 4.6. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\ \psi_2 \in \Psi(Y)$. If f is almost $(\psi, \psi_1\psi_2)$-continuous, then it is almost $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous.

Proof. Let $G \in g_{\psi_1}(f(x))$ for $x \in X$. Then there exists $V \in \psi_1(f(x))$ such that $V \subseteq G$. Since $G \in g_{\psi_1}(f(x))$, $f(x) \in G \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(G))$ and $x \in f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(G)))$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. So $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V)) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(G))$, $U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(G)))$ and $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(G))) \in g_\psi$. For $H = f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(G)))$, by Theorem 2.17 and Lemma 2.18, we have

$$f(H) = f(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(G)))) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(G)) \subseteq \iota_{\psi_1}(c_{\psi_2}(G)) = i_{\psi_1}(c_{\psi_2}(G)).$$

Hence, f is almost $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous.

The following example shows that the converse of Theorem 4.6 is not true.

Example 4.7. Let $X = \{a, b, c\}$. Consider generalized neighborhood systems ψ, ψ_1 and ψ_2 defined as follows:

$$\psi(a) = \{X\},\ \psi(b) = \{X\},\ \psi(c) = \{X\}.$$

$$\psi_1(a) = \{\{a\}\},\ \psi_1(b) = \{\{b, c\}\},\ \psi_1(c) = \{X\}.$$

$$\psi_2(a) = \{\{a\}, \{a, b\}\},\ \psi_2(b) = \{\{b, c\}\},\ \psi_2(c) = \{X\}.$$

Define an identity function $f : X \to X$. Then we get the following GT's g_ψ, g_{ψ_1} and g_{ψ_2} induced by ψ, ψ_1 and ψ_2, respectively:

$$g_\psi = \{\emptyset, X\},\ g_{\psi_1} = \{\emptyset, \{a\}, X\},\ g_{\psi_2} = \{\emptyset, \{a\}, X\}.$$

Then f is almost $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous. But, f is not almost $(\psi, \psi_1\psi_2)$-continuous at point a, and so f is not almost $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.8. Let $f : X \to Y$ be a function and $\psi \in \Psi(X)$. If f is almost $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous and $\psi_1 = \psi_{g_1}$, $\psi_2 = \psi_{g_2}$ for some GT's g_1 and g_2 on Y respectively, then f is almost $(\psi, \psi_1\psi_2)$-continuous.

Proof. Let $V \in \psi_1(f(x))$ for $x \in X$. By $\psi_1 = \psi_{g_1}$, $\psi_2 = \psi_{g_2}$ and Lemma 2.20, $g_1 = g_{\psi_1}$ and $g_2 = g_{\psi_2}$. So $V \in \psi_{g_1}(f(x))$ and $f(x) \in V \in g_1 = g_{\psi_1}$. By hypothesis, there exists $U \in g_\psi$ such that $x \in U$ and $f(U) \subseteq i_{g_{\psi_1}}(c_{g_{\psi_2}}(V))$. Since $U \in g_\psi$, there exists $H \in \psi(x)$ such that $H \subseteq U$. So by Theorem 2.17 and Lemma 2.18, we have

$$f(H) \subseteq f(U) \subseteq i_{g_{\psi_1}}(c_{g_{\psi_2}}(V)) \subseteq \iota_{\psi_1}(c_{g_{\psi_2}}(V)) = \iota_{\psi_1}(\gamma_{\psi_2}(V)).$$

Hence, f is almost $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.9. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is almost $(\psi, \psi_1\psi_2)$-continuous.
(2) $f^{-1}(\iota_{\psi_1}(B)) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every subset B of Y.
(3) $\gamma_\psi(f^{-1}(\gamma_{\psi_1}(\iota_{\psi_2}(B)))) \subseteq f^{-1}(\gamma_{\psi_1}(B))$ for every subset B of Y.

Proof. (1) ⇒ (2): Let $x \in f^{-1}(\iota_{\psi_1}(B))$. Then there exists $V \in \psi_1(f(x))$ such that $V \subseteq B$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. This implies

$$f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V)) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(B))$$

and

$$U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))).$$

Hence we have $x \in \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$.

(2) ⇒ (1): Let $V \in \psi_1(f(x))$ for $x \in X$. Since $f(x) \in V$, $x \in f^{-1}(\iota_{\psi_1}(V)) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(V))))$. Then there exists $U \in \psi(x)$ such that $U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(V)))$. So $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$ for $U \in \psi(x)$ and f is almost $(\psi, \psi_1\psi_2)$-continuous.

(2) ⇔ (3): Obvious.

Definition 4.10. Let $\psi_1, \psi_2 \in \Psi(X)$. Then A is said to be:

(1) (ψ_1, ψ_2)-semiopen if $A \subseteq \gamma_{\psi_2}(\iota_{\psi_1}(A))$.
(2) (ψ_1, ψ_2)-preopen if $A \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(A))$.
(3) (ψ_1, ψ_2)-β'-open if $A \subseteq \gamma_{\psi_2}(\iota_{\psi_1}(\gamma_{\psi_2}(A)))$.
(4) $r(\psi_1, \psi_2)$-open (or (ψ_1, ψ_2)-regular open) (resp. $r(\psi_1, \psi_2)$-closed) (or (ψ_1, ψ_2)-regular closed) if $A = \iota_{\psi_1}(\gamma_{\psi_2}(A))$ (resp. $A = \gamma_{\psi_1}(\iota_{\psi_2}(A))$).

The complement of (ψ_1,ψ_2)-semiopen (resp. (ψ_1,ψ_2)-preopen, (ψ_1,ψ_2)-β'-open) set is called (ψ_1,ψ_2)-semiclosed (resp. (ψ_1,ψ_2) -preclosed, (ψ_1,ψ_2)-β'-closed) set.

Remark 4.11. Every $r(\psi_1,\psi_2)$-open set is (ψ_1,ψ_2)-preopen and every (ψ_1,ψ_2)-preopen set is (ψ_1,ψ_2)-β'-open. Also every (ψ_1,ψ_2)-semiopen set is (ψ_1,ψ_2)-β'-open.

The above relations among them can be presented in the following diagram.

$$r(\psi_1,\psi_2)\text{-open} \Rightarrow (\psi_1,\psi_2)\text{-preopen} \Rightarrow (\psi_1,\psi_2)\text{-}\beta'\text{-open} \Leftarrow (\psi_1,\psi_2)\text{-semiopen}$$

Definition 4.12. Let $\psi_1,\psi_2 \in \Psi(X)$. Then A is said to be:

(1) $(\psi_1,\psi_2)'$-semiopen if $A \subseteq \gamma_{\psi_1}(\iota_{\psi_2}(A))$.
(2) $(\psi_1,\psi_2)'$-preopen if $A \subseteq \iota_{\psi_2}(\gamma_{\psi_1}(A))$.
(3) $(\psi_1,\psi_2)'$-β'-open if $A \subseteq \gamma_{\psi_1}(\iota_{\psi_2}(\gamma_{\psi_1}(A)))$.

The complement of $(\psi_1,\psi_2)'$-semiopen (resp. $(\psi_1,\psi_2)'$-preopen, $(\psi_1,\psi_2)'$-β'-open) set is called $(\psi_1,\psi_2)'$-semiclosed (resp. $(\psi_1,\psi_2)'$ -preclosed, $(\psi_1,\psi_2)'$-β'-closed) set.

Remark 4.13. Every $(\psi_1,\psi_2)'$-preopen set is (ψ_1,ψ_2)-β'-open and also (ψ_1,ψ_2)- semiopen is (ψ_1,ψ_2)-β'-open. We have the following diagram.

$$(\psi_1,\psi_2)'\text{-preopen} \Rightarrow (\psi_1,\psi_2)\text{-}\beta'\text{-open} \Leftarrow (\psi_1,\psi_2)\text{-semiopen.}$$

Theorem 4.14. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is almost $(\psi,\psi_1\psi_2)$-continuous.
(2) $f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every $B \in \psi_1(f(x))$.
(3) $\iota_\psi(f^{-1}(B)) = f^{-1}(B)$ for every $r(\psi_1,\psi_2)$-open $B \in \psi_1(f(x))$.
(4) $\iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))) = f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$ for every $B \in \psi_1(f(x))$.

Proof. (1) $\Rightarrow$ (2): Let $B \in \psi_1(f(x))$. Then $x \in f^{-1}(B)$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(B))$. This implies $U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$ for $U \in \psi(x)$. Hence we have $x \in \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$.

(2) $\Rightarrow$ (3): Let $B \in \psi_1(f(x))$ is $r(\psi_1,\psi_2)$-open. By hypothesis, $f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))) = \iota_\psi(f^{-1}(B))$. Then we have $\iota_\psi(f^{-1}(B)) = f^{-1}(B)$.

(3) $\Rightarrow$ (4): $\iota_{\psi_1}(\gamma_{\psi_2}(B))$ is $r(\psi_1,\psi_2)$-open for $B \in \psi_1(f(x))$, it is obvious.

(4) $\Rightarrow$ (1): Let $B \in \psi_1(f(x))$. By hypothesis, $\iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))) = f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. This implies $x \in f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))) = f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. Then there exists $U \in \psi(x)$ such that $U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. Hence $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(B))$ for $U \in \psi(x)$ and f is almost $(\psi,\psi_1\psi_2)$-continuous.

Theorem 4.15. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is almost $(\psi,\psi_1\psi_2)$-continuous.
(2) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every subset B of Y.
(3) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every (ψ_1,ψ_2)-β'-open subset B of Y.
(4) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every (ψ_1,ψ_2)-semiopen subset B of Y.
(5) $f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every (ψ_1,ψ_2)-preopen subset B of Y.
(6) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every $B \in \psi_1(f(x))$.

Proof. (1) $\Rightarrow$ (2): Let $B \subseteq Y$ and $x \in f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. Then there exists $U \in \psi_1(f(x))$ such that $U \subseteq \gamma_{\psi_2}(B)$. By hypothesis, there exists $G \in \psi(x)$ such that $f(G) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(U))$. This implies

$$f(G) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(U)) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(\gamma_{\psi_2}(B))) = \iota_{\psi_1}(\gamma_{\psi_2}(B)).$$

Hence we have $x \in \iota_{\psi}(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Since every (ψ_1, ψ_2)-semiopen set is (ψ_1, ψ_2)-β'-open, it is obvious.

(4) $\Rightarrow$ (5): Let B is (ψ_1, ψ_2)-preopen subset of Y. Then $B \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(B))$ and $\gamma_{\psi_2}(B) \subseteq \gamma_{\psi_2}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. So we know that $\gamma_{\psi_2}(B)$ is (ψ_1, ψ_2) -semiopen set. By hypothesis,

$$\begin{aligned} f^{-1}(B) &\subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) = f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(\gamma_{\psi_2}(B)))) \\ &\subseteq \iota_{\psi}(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(\gamma_{\psi_2}(B))))) \\ &= \iota_{\psi}(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))). \end{aligned}$$

(5) $\Rightarrow$ (6): Let $B \in \psi_1(f(x))$. Since every $\iota_{\psi_1}(\gamma_{\psi_2}(B))$ is $r(\psi_1, \psi_2)$-open set, it is (ψ_1, ψ_2)-preopen set. Hence it is obvious.

(6) $\Rightarrow$ (1): By Theorem 4.14, it is obvious.

Theorem 4.16. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\ \psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is almost $(\psi, \psi_1\psi_2)$-continuous.
(2) $\gamma_{\psi}(f^{-1}(B)) \subseteq f^{-1}(\gamma_{\psi_1}(B))$ for every $r(\psi_1, \psi_2)$-closed subset B of Y.
(3) $\gamma_{\psi}(f^{-1}(\gamma_{\psi_1}(B))) \subseteq f^{-1}(\gamma_{\psi_1}(B))$ for every $(\psi_1, \psi_2)'$-β'-open subset B of Y.
(4) $\gamma_{\psi}(f^{-1}(\gamma_{\psi_1}(B))) \subseteq f^{-1}(\gamma_{\psi_1}(B))$ for every $(\psi_1, \psi_2)'$-semiopen subset B of Y.
(5) $\gamma_{\psi}(f^{-1}(\gamma_{\psi_1}(\iota_{\psi_2}(B)))) \subseteq f^{-1}(\gamma_{\psi_1}(\iota_{\psi_2}(B)))$ for every subset B of Y.
(6) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_{\psi}(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let B is $r(\psi_1, \psi_2)$-closed subset of Y. Then $B = \gamma_{\psi_1}(\iota_{\psi_2}(B))$. By Theorem 4.9, we have $\gamma_{\psi}(f^{-1}(B)) \subseteq f^{-1}(\gamma_{\psi_1}(B))$.

(2) $\Rightarrow$ (3): Let B is $(\psi_1, \psi_2)'$-β'-open subset of Y. Then $\gamma_{\psi_1}(B) = \gamma_{\psi_1}(\iota_{\psi_2}(\gamma_{\psi_1}(B)))$, so we know that $\gamma_{\psi_1}(B)$ is $r(\psi_1, \psi_2)$-closed. By hypothesis, $\gamma_{\psi}(f^{-1}(\gamma_{\psi_1}(B))) \subseteq f^{-1}(\gamma_{\psi_1}(B))$.

(3) $\Rightarrow$ (4): Since every $(\psi_1, \psi_2)'$-semiopen set is $(\psi_1, \psi_2)'$-β'-open, it is obvious.

(4) $\Rightarrow$ (5): Let $B \subseteq Y$. Since $\gamma_{\psi_1}(\iota_{\psi_2}(B))$ is $(\psi_1, \psi_2)'$-semiopen set, it is obvious.

(5) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (1): Let $B \in \psi_1(f(x))$. By hypothesis,

$$x \in f^{-1}(B) \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_{\psi}(f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))).$$

Then there exists $U \in \psi(x)$ such that $U \subseteq f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B)))$. This implies $f(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(B))$. Hence we have f is almost $(\psi, \psi_1\psi_2)$-continuous.

Definition 4.17. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. Then f is said to be mixed weakly $(\psi, \psi_1\psi_2)$-continuous at $x \in X$ if for $x \in X$ and $V \in \psi_1(f(x))$, there exists $U \in \psi(x)$ such that $f(U) \subseteq \gamma_{\psi_2}(V)$. Then f is said to be mixed weakly $(\psi, \psi_1\psi_2)$-continuous (briefly; weakly $(\psi, \psi_1\psi_2)$-continuous) if it is mixed weakly $(\psi, \psi_1\psi_2)$-continuous at every point of X.

Remark 4.18. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. If f is almost $(\psi, \psi_1\psi_2)$-continuous, then it is weakly $(\psi, \psi_1\psi_2)$-continuous. But the converse is not always true as shown in the next example.

Example 4.19. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. Consider generalized neighborhood systems $\psi \in \Psi(X)$ and $\psi_1, \psi_2 \in \Psi(Y)$ defined as follows:

$$\psi(a) = \{\{a\}\},\ \psi(b) = \{\{a, b\}\},\ \psi(c) = \{\{a, c\}\}.$$

$$\psi_1(1) = \{\{1, 2\}\},\ \psi_1(2) = \{\{2\}\},\ \psi_1(3) = \{\{1, 3\}\}.$$

$$\psi_2(1) = \{\{1, 2\}, Y\},\ \psi_2(2) = \{\{2\}, Y\},\ \psi_2(3) = \{\{1, 3\}, Y\}.$$

A function $f : X \to Y$ defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 3$. Then f is weakly $(\psi, \psi_1\psi_2)$-continuous. But, f is not almost $(\psi, \psi_1\psi_2)$-continuous at point c, and so f is not almost $(\psi, \psi_1\psi_2)$-continuous.

Definition 4.20. Let $\psi_1, \psi_2 \in \Psi(X)$ and $x \in X$. Then ψ_1 is ψ_2-EDC iff $V \in \psi_1(x)$ implies that $\gamma_{\psi_2}(V) \in \psi_1(x)$.

Theorem 4.21. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1, \psi_2 \in \Psi(Y)$. If ψ_1 is ψ_2-EDC, then the following are equivalent:

(1) f is almost $(\psi, \psi_1\psi_2)$-continuous.
(2) f is weakly $(\psi, \psi_1\psi_2)$-continuous.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): Let $V \in \psi_1(x)$ for $x \in X$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \gamma_{\psi_2}(V)$. Since ψ_1 is ψ_2-EDC, $\gamma_{\psi_2}(V) \in \psi_1(x)$ and $\iota_{\psi_1}(\gamma_{\psi_2}(V)) = \gamma_{\psi_2}$. This implies $f(U) \subseteq \gamma_{\psi_2}(V) = \iota_{\psi_1}(\gamma_{\psi_2}(V))$. Hence f is almost $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.22. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. If f is weakly $(\psi, \psi_1\psi_2)$-continuous, then it is weakly $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous.

Proof. Let $G \in g_{\psi_1}(f(x))$. Then there exists $V \in \psi_1(f(x))$ such that $V \subseteq G$. Since $G \in g_{\psi_1}(f(x))$, $f(x) \in G \subseteq \gamma_{\psi_2}(G)$ and $x \in f^{-1}(G) \subseteq f^{-1}(\gamma_{\psi_2}(G))$. By hypothesis, there exists $U \in \psi(x)$ such that $f(U) \subseteq \gamma_{\psi_2}(V)$. So $f(U) \subseteq \gamma_{\psi_2}(V) \subseteq \gamma_{\psi_2}(G)$, $U \subseteq f^{-1}(\gamma_{\psi_2}(V)) \subseteq f^{-1}(\gamma_{\psi_2}(G))$ and $f^{-1}(\gamma_{\psi_2}(G)) \in g_\psi$. For $H = f^{-1}(\gamma_{\psi_2}(G))$, $f(H) = f(f^{-1}(\gamma_{\psi_2}(G))) \subseteq \gamma_{\psi_2}(G) \subseteq c_{g_{\psi_2}}(G)$. Hence, f is weakly $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous.

The following example shows that the converse of Theorem 4.22 is not true.

Example 4.23. Let $X = \{a, b, c\}$ and ψ, ψ_1, $\psi_2 \in \Psi(X)$ be generalized neighborhood systems defined as in Example 4.7. Define an identity function $f : X \to X$. Then f is weakly $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous. But, f is not weakly $(\psi, \psi_1\psi_2)$-continuous at point a, and so f is not weakly $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.24. Let $f : X \to Y$ be a function and $\psi \in \Psi(X)$. If f is weakly $(g_\psi, g_{\psi_1}g_{\psi_2})$-continuous and $\psi_1 = \psi_{g_1}$, $\psi_2 = \psi_{g_2}$ for some GT's g_1 and g_2 on Y respectively, then f is weakly $(\psi, \psi_1\psi_2)$-continuous.

Proof. Let $V \in \psi_1(f(x))$ for $x \in X$. Since $\psi_1 = \psi_{g_1}$, $\psi_2 = \psi_{g_2}$ and Lemma 2.20, $g_1 = g_{\psi_1}$ and $g_2 = g_{\psi_2}$. Then $V \in \psi_{g_1}(f(x))$ and $f(x) \in V \in g_1 = g_{\psi_1}$. By hypothesis, there exists $U \in g_\psi$ containing x such that $f(U) \subseteq c_{g_{\psi_2}}(V)$. According to definition of $U \in g_\psi$, there exists $H \in \psi(x)$ such that $H \subseteq U$. So by Lemma 2.18., $f(H) \subseteq \gamma_{\psi_2}(V)$. Hence, f is weakly $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.25. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and $\psi_1,\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is weakly $(\psi, \psi_1\psi_2)$-continuous.
(2) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every subset B of Y.
(3) $\gamma_\psi(f^{-1}(\iota_{\psi_2}(B))) \subseteq f^{-1}(\gamma_{\psi_1}(\iota_{\psi_2}(B)))$ for every subset B of Y.

Proof. It is proved as in Theorem 4.9.

Theorem 4.26. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and ψ_1, $\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is weakly $(\psi, \psi_1\psi_2)$-continuous.
(2) $\gamma_\psi(f^{-1}(\iota_{\psi_2}(B)) \subseteq f^{-1}(B)$ for every $r(\psi_1, \psi_2)$-closed subset B of Y.
(3) $f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B))$ for every $r(\psi_1, \psi_2)$-open subset B of Y.
(4) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every $B \in \psi_1(f(x))$.

Proof. (1) $\Rightarrow$ (2): Let B is $r(\psi_1, \psi_2)$-closed subset of Y. Then from Theorem 4.25 (3), we have $\gamma_\psi(f^{-1}(\iota_{\psi_2}(B)) \subseteq f^{-1}(B)$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Let $B \in \psi_1(f(x))$. Since $\iota_{\psi_1}(\gamma_{\psi_2}(B))$ is $r(\psi_1, \psi_2)$-open, it is obvious.

(4) $\Rightarrow$ (1): Let $B \in \psi_1(f(x))$. Hence by Theorem 4.25 (2), f is weakly $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.27. Let $f : X \to Y$ be a function, $\psi \in \Psi(X)$ and ψ_1, $\psi_2 \in \Psi(Y)$. Then the following are equivalent:

(1) f is weakly $(\psi, \psi_1\psi_2)$-continuous.
(2) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every subset B of Y.
(3) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every (ψ_1, ψ_2)-β'- open subset B of Y.
(4) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every (ψ_1, ψ_2)-semiopen subset B of Y.
(5) $f^{-1}(B) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every (ψ_1, ψ_2)-preopen B of Y.
(6) $f^{-1}(\iota_{\psi_1}(\gamma_{\psi_2}(B))) \subseteq \iota_\psi(f^{-1}(\gamma_{\psi_2}(B)))$ for every $B \in \psi_1(f(x))$.

Proof. It is proved as in Theorem 4.15.

Theorem 4.28. Let $\psi \in \Psi(X)$, $\psi' \in \Psi(Y)$ and $\psi_1, \psi_2 \in \Psi(Z)$. If $f : X \to Y$ is (ψ, ψ')- continuous and $g : Y \to Z$ is almost $(\psi', \psi_1\psi_2)$-continuous, then the composition $g \circ f : X \to Z$ is almost $(\psi, \psi_1\psi_2)$-continuous.

Proof. Let $x \in X$ and $V \in \psi_1(g(f(x)))$. Since g is almost $(\psi', \psi_1\psi_2)$-continuous, there exists $U \in \psi'(f(x))$ such that $g(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. Since f is (ψ, ψ')-continuous, there exists $G \in \psi(x)$ such that $f(G) \subseteq U$. For $G \in \psi(x)$, $(g \circ f)(G) \subseteq g(U) \subseteq \iota_{\psi_1}(\gamma_{\psi_2}(V))$. Thus, $g \circ f$ is almost $(\psi, \psi_1\psi_2)$-continuous.

Theorem 4.29. Let $\psi \in \Psi(X)$, $\psi' \in \Psi(Y)$ and $\psi_1, \psi_2 \in \Psi(Z)$. If $f : X \to Y$ is (ψ, ψ')- continuous and $g : Y \to Z$ is weakly $(\psi', \psi_1\psi_2)$-continuous, then the composition $g \circ f : X \to Z$ is weakly $(\psi, \psi_1\psi_2)$-continuous.

Proof. Let $x \in X$ and $V \in \psi_1(g(f(x)))$. Since g is weakly $(\psi', \psi_1\psi_2)$-continuous, there exists $U \in \psi'(f(x))$ such that $g(U) \subseteq \gamma_{\psi_2}(V)$. Since f is (ψ, ψ')-continuous, there exists $G \in \psi(x)$ such that $f(G) \subseteq U$. For $G \in \psi(x)$, $(g \circ f)(G) \subseteq g(U) \subseteq \gamma_{\psi_2}(V)$. Thus, $g \circ f$ is weakly $(\psi, \psi_1\psi_2)$-continuous.

The following diagram summarizes the relations that exist among almost $(\psi, \psi_1\psi_2)$-continuity, weakly $(\psi, \psi_1\psi_2)$- continuity, almost $(g_\psi, g_{\psi_1} g_{\psi_2})$-continuity and some other variants of continuity.

$$\begin{array}{ccccc}
(\psi,\psi_1)\text{-continuity} & \longrightarrow & \text{almost } (\psi,\psi_1\psi_2)\text{-continuity} & \longrightarrow & \text{weakly } (\psi,\psi_1\psi_2)\text{-continuity} \\
\downarrow & & \downarrow & & \downarrow \\
(g_\psi, g_{\psi_1})\text{-continuity} & \longrightarrow & \text{almost } (g_\psi, g_{\psi_1} g_{\psi_2})\text{-continuity} & \longrightarrow & \text{weakly } (g_\psi, g_{\psi_1} g_{\psi_2})\text{-continuity}
\end{array}$$

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