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Article

Magnetic Curves in Homothetic *s*-th Sasakian Manifolds

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Abstract: We investigate normal magnetic curves in $(2n+s)$-dimensional homothetic s-th Sasakian manifolds as a generalization of S-manifolds. We show that a curve γ is a normal magnetic curve in a homothetic s-th Sasakian manifold if and only if its osculating order satisfies $r \leq 3$ and it belongs to a family of θ_i-slant helices. Additionally, we construct a homothetic s-th Sasakian manifold using generalized D-homothetic transformations and present the parametric equations of normal magnetic curves in this manifold.

Keywords: magnetic curve; θ_i-slant curve; homothetic s-th Sasakian manifold

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1. Introduction

Magnetic curves are trajectories of charged particles in a Riemannian or pseudo-Riemannian manifold (M, g) under the influence of a magnetic field. Formally, they are critical points of a variational problem defined by a magnetic Lagrangian, which incorporates both the kinetic energy and the interaction with the magnetic field [1]. Let (M, g) be a Riemannian manifold, and let F be a closed 2-form representing the *magnetic field*. A curve $\gamma : I \to M$ is called a *magnetic curve* if it satisfies the following equation:

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{1}$$

where ∇ is the Levi-Civita connection of g and $\Phi : \chi(M) \to \chi(M)$ is a $(1,1)$-type tensor field determined by F, defined as [2–4]:

$$g(\Phi U, V) = F(U, V), \ \ \forall U, V \in \chi(M). \tag{2}$$

In Equation (1), the left-hand term, $\nabla_{\gamma'}\gamma'$, represents the geodesic acceleration and the right-hand term, $\Phi(\gamma')$, represents the *Lorentz force* associated to the magnetic field. The equation itself is well-known as *the Lorentz equation*.

Magnetic curves generalize the concept of geodesics to include the effect of a magnetic field [5]. When $F = 0$, the equation reduces to the geodesic equation, and γ describes free motion in the manifold. Nonzero F introduces a deviation due to the magnetic force. From the preservation of energy, the speed $\|\gamma'\|$ of a magnetic curve is constant because the magnetic force does no work [1]:

$$\frac{d}{dt}\frac{1}{2}\|\gamma'\|^2 = g(\nabla_{\gamma'}\gamma', \gamma') = g(\Phi(\gamma'), \gamma') = F(\gamma', \gamma') = 0.$$

Moreover, if $\|\gamma'\| = 1$, γ is called a *normal magnetic curve* [6].

In ref. [7], Nakagawa introduced the concept of framed f-structures, extending the idea of almost contact structures. Later, in ref. [8], Hasegawa, Okuyama, and Abe defined the notion of p-th Sasakian manifolds, providing illustrative examples to deepen the understanding of these structures. In [9], Alegre, Fernandez, and Prieto-Martin introduced a new class of metric f-manifolds, expanding the study of almost contact metric structures. Their work explored the foundational properties of these manifolds and provided several examples to demonstrate their geometric significance.

Subsequently, Adachi [2] explored the bounds of curvature and the behavior of magnetic trajectories on Hadamard surfaces. His findings revealed that under certain curvature constraints, every normal trajectory in a 2-dimensional, complete, and simply connected Riemannian manifold extends unboundedly in both directions.

In the realm of contact geometry, Baikoussis and Blair [10] investigated Legendre curves in 3-dimensional contact manifolds, demonstrating that the torsion of these curves is invariably 1 in Sasakian manifolds. Building on these foundations, Cho, Inoguchi, and Lee [11] defined and studied slant curves. Extending this idea, the first author introduced θ_i-slant curves in S-manifolds [12], broadening the framework with innovative examples and applications in framed metric f-structures.

Cabrerizo, Fernandez, and Gomez [13] developed an elegant approach for constructing almost contact metric structures compatible with given metrics on 3-dimensional oriented Riemannian manifolds. Subsequently, Druta-Romaniuc et al. [6] investigated magnetic trajectories in Sasakian $(2n+1)$-manifolds under contact magnetic fields $F_q = q\omega$, where ω is the fundamental 2-form. Their research paved the way for further explorations, such as particle trajectories in cosymplectic manifolds [14] and closed magnetic paths on 3-dimensional Berger spheres [15]. In ref. [16], Jleli, Munteanu, and Nistor advanced these studies by examining magnetic curves in almost contact metric manifolds and concluded that normal magnetic curves correspond to helices of order 5 or less.

In para-Kaehler manifolds, Jleli and Munteanu [17] analyzed spacelike and timelike normal magnetic trajectories associated to para-Kaehler 2-forms, establishing their circular nature. Their earlier works [18,19] provided classifications of unit-speed Killing magnetic curves and examined normal magnetic trajectories on Sasakian spheres S^{2n+1}, showing their restriction to totally geodesic subspheres S^3. This line of research culminated in a study of closed normal trajectories on 3-dimensional tori derived from various contact forms on $\mathbb{E}^3$ [20].

Further developments included the introduction of T-magnetic, N-magnetic, and B-magnetic curves in 3-dimensional semi-Riemannian manifolds [21], as well as the classification of magnetic trajectories generated by Killing vector fields in normal paracontact metric 3-manifolds [22]. The second author also contributed by examining magnetic curves in the 3-dimensional Heisenberg group [23]. More recently, the present authors focused on slant magnetic curves in S-manifolds [24], delving into their geometric characteristics under specific magnetic influences. These contributions highlight the interplay between curvatures and contact structures in shaping the behavior of magnetic curves. For a deeper understanding of the foundational concepts underlying these advancements, readers are encouraged to consult [25–27].

Motivated by recent studies, this paper investigates normal magnetic curves within the context of $(2n+s)$-dimensional homothetic s-th Sasakian manifolds, which serve as a generalization of S-manifolds. We obtain that a curve γ qualifies as a normal magnetic curve in a homothetic s-th Sasakian manifold if and only if its osculating order satisfies $r \leq 3$ and it belongs to a family of θ_i-slant helices. Moreover, we construct a homothetic s-th Sasakian manifold using generalized D-homothetic transformations and provide the parametric equations describing normal magnetic curves within this manifold.

2. Preliminaries

A Riemannian manifold (M, g) is called a *homothetic s-th Sasakian manifold* if it satisfies the following properties for all $U, V \in \chi(M)$:

$$f^2U = -U + \sum_{i=1}^{s} \eta_i(U)\xi_i, \tag{3}$$

$$\eta_i(\xi_j) = \delta_{ij},\ f\xi_i = 0,\ \eta_i(fU) = 0,\ \eta_i(U) = g(U, \xi_i),$$

$$g(fU, fV) = g(U, V) - \sum_{i=1}^{s} \eta_i(U)\eta_i(V), \tag{4}$$

$$d\eta_i(U, V) = -d\eta_i(V, U) = \alpha_i g(U, fV),$$

where f is a $(1, 1)$-type tensor field, ξ_i $(i = 1, 2, \ldots, s)$ are Killing characteristic vector fields, η_i $(j = 1, 2, \ldots, s)$ are 1-forms, α_i $(i = 1, 2, \ldots, s)$ are nonzero constants, and this f*-structure* is normal [8]. It is denoted in short by $M = (M^{2n+s}, f, \xi_i, \eta_i, g)$. If $\alpha_i = 1$ $(i = 1, 2, \ldots, s)$, then M becomes an S-manifold. It is important to mention that these manifolds are a subclass of trans-S-manifolds [9]. In a homothetic s-th Sasakian manifold, we have

$$(\nabla_U f)V = \sum_{i=1}^{s} \alpha_i \Big\{ g(fU, fV)\xi_i + \eta_i(V) f^2 U \Big\},$$

and

$$\nabla_U \xi_i = -\alpha_i fU, i \in \{1, \ldots, s\},$$

where ∇ is the Levi-Civita connection associated to g. The *fundamental* 2*-form* on M is given by

$$\omega(U, V) = g(U, fV). \tag{5}$$

One can easily show that ω is closed because

$$d\omega = d\left(\frac{1}{\alpha_i} d\eta_i\right) = \frac{1}{\alpha_i} d^2\eta_i = 0.$$

As a result, we can define the *magnetic field* F_q *with strength* q as

$$F_q(U, V) = q\omega(U, V),$$

where $U, V \in \chi(M)$, and q is a real constant [16]. By using Equations (2) and (5), the Lorentz force Φ is calculated as

$$\Phi_q = -qf.$$

Consequently, we can rewrite the Lorentz equation in (1) as

$$\nabla_T T = -qfT, \tag{6}$$

where $\gamma : I \to M$ is an arc-length parameterized smooth curve, and $T = \gamma'$ (see [6,16]).

3. Main Results

Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a smooth curve. Then, the set of vector fields $\{T = E_1,\ E_2, \ldots, E_r\}$ is called *the Frenet frame field of* γ, which satisfies

$$\begin{aligned}
T &= E_1 = \gamma', \\
\nabla_T T &= \kappa_1 E_2, \\
\nabla_T E_2 &= -\kappa_1 T + \kappa_2 E_3, \\
&\dots \\
\nabla_T E_j &= -\kappa_{j-1} E_{j-1} + \kappa_j E_{j+1},\ (2 < j < r), \\
&\dots \\
\nabla_T E_r &= -\kappa_{r-1} E_{r-1},
\end{aligned} \tag{7}$$

where ∇ denotes the Levi-Civita connection. In this case, we call the positive integer $r \leq n$ *the osculating order* and $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ *the curvatures of* γ. Consequently, γ is called a *Frenet curve of osculating order* r.

Curves are classified depending on their curvatures as follows: A Frenet curve of osculating order $r = 1$ is a *geodesic*. A Frenet curve of osculating order $r = 2$ with constant curvature κ_1 is a *circle*. A Frenet curve of osculating order $r \leq n$ with constant curvatures $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ is a *helix of order* r. We call a helix of order $r = 3$ shortly as *helix*.

Let $M = (M^{2n+s}, f, \xi_i, \eta_i, g)$ be a homothetic s-th Sasakian manifold and $\gamma : I \to M$ a unit-speed curve. We call the functions $\theta_i = \theta_i(t)$ *the contact angles* between T and ξ_i, that is,

$$\cos\theta_i(t) = g(T, \xi_i).$$

γ is called a θ_i*-slant curve* if all θ_i are constants. If these constant contact angles are all equal to the same value, we call γ a *slant curve*. Additionally, if the contact angles are all equal to $\frac{\pi}{2}$, then it is called a *Legendre curve* and it becomes a 1-dimensional integral submanifold of the contact distribution (see [12]).

For a θ_i-slant curve of osculating order r in a homothetic s-th Sasakian manifold, the following calculations are direct:

$$\begin{aligned}
\nabla_T f T &= (\nabla_T f)T + f\nabla_T T \\
&= \left(1 - \sum_{i=1}^{s} \cos^2\theta_i\right)\left(\sum_{i=1}^{s} \alpha_i \xi_i\right) \\
&\quad + \left(\sum_{i=1}^{s} \alpha_i \cos\theta_i\right)\left(-T + \sum_{i=1}^{s} \cos\theta_i \xi_i\right) + \kappa_1 f E_2
\end{aligned}$$

and

$$\nabla_T \xi_i = -\alpha_i f T,\ i = 1, 2, \dots, s.$$

By differentiating $\eta_i(T) = \cos\theta_i$, we find

$$\eta_i(E_2) = 0,\ i = 1, 2, \dots, s.$$

Now we can state our first proposition, constructing a one-way bridge from normal magnetic curves to θ_i-slant curves:

Proposition 1. *Let* $(M^{2n+s}, f, \xi_i, \eta_i, g)$ *be a homothetic s-th Sasakian manifold and consider the contact magnetic field* F_q *for* $q \neq 0$*. If* $\gamma : I \to M$ *is a normal magnetic curve associated to* F_q *in M, then its contact angles are constants, i.e.,* γ *is a* θ_i*-slant curve.*

Proof. Let $\gamma : I \to M$ be a normal magnetic curve associated to F_q in M. Then, using

$$\nabla_T T = -qfT,$$

we obtain

$$\begin{aligned} g(\nabla_T T, \xi_i) &= g(-qfT, \xi_i) = 0 \\ &= \frac{d}{dt} g(T, \xi_i) - g(T, \nabla_T \xi_i) \\ &= \frac{d}{dt} g(T, \xi_i) - g(T, -\alpha_i fT) \\ &= \frac{d}{dt} g(T, \xi_i). \end{aligned}$$

As a result, we have

$$g(T, \xi_i) = \eta_i(T) = \cos\theta_i = \text{constant}.$$

□

After this proposition, we can present the following theorem, which is the main theorem of the paper.

Theorem 1. *Let $(M^{2n+s}, f, \xi_i, \eta_i, g)$ be a homothetic s-th Sasakian manifold and consider the contact magnetic field F_q for $q \neq 0$. Then γ is a normal magnetic curve associated to F_q in M if and only if γ belongs to the following list:*

(a) *non-Legendre θ_i-slant geodesics as integral curves of $\sum_{i=1}^{s} \cos\theta_i \xi_i$, where $\sum_{i=1}^{s} \cos^2\theta_i = 1$;*

(b) *non-Legendre θ_i-slant circles with the curvature*

$$\kappa_1 = \sqrt{q^2 - \sum_{i=1}^{s} \alpha_i^2},$$

having contact angles

$$\theta_i = \arccos\left(\frac{\alpha_i}{q}\right),\ i = 1, 2, \ldots, s,$$

and the Frenet frame field

$$\left\{ T, \frac{-qfT}{\sqrt{q^2 - \sum_{i=1}^{s} \alpha_i^2}} \right\},$$

where $|q| > \sqrt{\sum_{i=1}^{s} \alpha_i^2}$;

(c) *Legendre helices with curvatures $\kappa_1 = |q|$ and $\kappa_2 = \sqrt{\sum_{i=1}^{s} \alpha_i^2}$, having the Frenet frame field*

$$\left\{ T, -sgn(q) fT, \frac{-sgn(q)}{\sqrt{\sum_{i=1}^{s} \alpha_i^2}} \sum_{i=1}^{s} \alpha_i \xi_i \right\};$$

(d) *θ_i-slant helices with*

$$\kappa_1 = |q|\sqrt{1 - \sum_{i=1}^{s} \cos^2 \theta_i},$$
$$\kappa_2 = \sqrt{\Lambda};$$

having the Frenet frame field

$$\left\{ T, \frac{-sgn(q)fT}{\sqrt{1 - \sum\limits_{i=1}^{s} \cos^2 \theta_i}}, E_3 \right\};$$

where we denote

$$\begin{aligned} \Lambda \;=\; & \left(\sum_{i=1}^{s} \cos^2 \theta_i\right) q^2 - 2\left(\sum_{i=1}^{s} \alpha_i \cos \theta_i\right) q + \left(\sum_{i=1}^{s} \alpha_i \cos \theta_i\right)^2 \\ & + \left(1 - \sum_{i=1}^{s} \cos^2 \theta_i\right)\left(\sum_{i=1}^{s} \alpha_i^2\right), \end{aligned} \tag{8}$$

$$E_3 = \frac{-sgn(q)}{\sqrt{\Lambda}\sqrt{1 - \sum\limits_{i=1}^{s} \cos^2 \theta_i}} \left[\begin{array}{c} \left(q \sum\limits_{i=1}^{s} \cos^2 \theta_i - \sum\limits_{i=1}^{s} \alpha_i \cos \theta_i\right) T + \left(1 - \sum\limits_{i=1}^{s} \cos^2 \theta_i\right)\left(\sum\limits_{i=1}^{s} \alpha_i \xi_i\right) \\ + \left(-q + \sum\limits_{i=1}^{s} \alpha_i \cos \theta_i\right)\left(\sum\limits_{i=1}^{s} \cos \theta_i \xi_i\right) \end{array} \right] \tag{9}$$

and $\sum\limits_{i=1}^{s} \cos^2 \theta_i < 1$.

Proof. Let γ be a normal magnetic curve for F_q in M. Then γ is a θ_i-slant curve with constant contact angles θ_i, $i = 1, 2, \ldots, s$. From Frenet equations and the Lorentz equation, we find

$$\nabla_T T = \kappa_1 E_2 = -qfT.$$

If $\kappa_1 = 0$, then $-qfT = 0$ gives us $fT = 0$. Thus, we find

$$f^2 T = -T + \sum_{i=1}^{s} \cos \theta_i \xi_1 = 0,$$

which results in

$$T = \sum_{i=1}^{s} \cos \theta_i \xi_1.$$

From $g(T, T) = 1$, we obtain $\sum\limits_{i=1}^{s} \cos^2 \theta_i = 1$. Hence, γ belongs to (a) in the list.

Let $\kappa_1 \neq 0$. In the expression $\kappa_1 E_2 = -qfT$, by taking the norm of both sides, we find

$$\begin{aligned} \kappa_1 \;&=\; \|-qfT\| = |q|.\|fT\| \\ &=\; |q|\sqrt{g(fT, fT)} \\ &=\; |q|\sqrt{g(T, T) - \sum_{i=1}^{s} [\eta_i(T)]^2} \\ &=\; |q|\sqrt{1 - \sum_{i=1}^{s} \cos^2 \theta_i}, \end{aligned} \tag{10}$$

which is a constant.

As a result, we can write

$$fT = -sgn(q)\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} E_2. \tag{11}$$

If $\kappa_2 = 0$, then γ is a circle, since it has constant κ_1. For θ_i-slant curves, differentiating $\eta_i(T) = \cos\theta_i$ =constant, we also have $\eta_i(E_2) = 0$. If we differentiate once again,

$$\begin{aligned}
\nabla_T \eta_i(E_2) &= 0 \\
&= g(\nabla_T E_2, \xi_i) + g(E_2, \nabla_T \xi_i) \\
&= g(-\kappa_1 T, \xi_i) + g(E_2, -\alpha_i fT) \\
&= -|q|\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i}\cos\theta_i + \alpha_i.sgn(q)\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} \\
&= sgn(q)\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i}(-q\cos\theta_i + \alpha_i).
\end{aligned}$$

Thus, we deduce that $-q\cos\theta_i + \alpha_i = 0$, or else $\sum\limits_{i=1}^{s} \cos^2\theta_i = 1$ would be the same as (a) in the list and γ would be a geodesic. This gives us

$$\theta_i = \arccos\left(\frac{\alpha_i}{q}\right),\ i = 1, 2, \ldots, s.$$

From the fact that $\alpha_i > 0$, γ cannot be Legendre. So, κ_1 becomes

$$\begin{aligned}
\kappa_1 &= |q|\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} \\
&= |q|\sqrt{1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{q^2}} \\
&= \sqrt{q^2 - \sum_{i=1}^{s} \alpha_i^2} > 0,
\end{aligned}$$

that is, $|q| > \sqrt{\sum\limits_{i=1}^{s} \alpha_i^2}$. In this case, γ belongs to (b) in the list.

Let $\kappa_2 \neq 0$. For θ_i-slant curves, we calculate

$$f^2 T = -T + \sum_{i=1}^{s} \cos\theta_i \xi_i \tag{12}$$

and

$$\begin{aligned}
\nabla_T fT = &\left(1 - \sum_{i=1}^{s} \cos^2\theta_i\right)\left(\sum_{i=1}^{s} \alpha_i \xi_i\right) \\
&+ \left(-q + \sum_{i=1}^{s} \alpha_i \cos\theta_i\right)\left(-T + \sum_{i=1}^{s} \cos\theta_i \xi_i\right).
\end{aligned} \tag{13}$$

If we differentiate (11), we also have

$$\begin{aligned}
\nabla_T fT &= -sgn(q)\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}\,\nabla_T E_2 \\
&= -sgn(q)\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}\,(-\kappa_1 T+\kappa_2 E_3) \\
&= q\left(1-\sum_{i=1}^{s}\cos^2\theta_i\right)T-\kappa_2 sgn(q)\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}\,E_3.
\end{aligned} \tag{14}$$

From Equations (13) and (14), it follows that

$$\begin{aligned}
-\kappa_2 sgn(q)\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}\,E_3 &= \left(q\sum_{i=1}^{s}\cos^2\theta_i-\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)T+\left(1-\sum_{i=1}^{s}\cos^2\theta_i\right)\left(\sum_{i=1}^{s}\alpha_i\xi_i\right) \\
&\quad+\left(-q+\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)\left(\sum_{i=1}^{s}\cos\theta_i\xi_i\right).
\end{aligned}$$

By taking the norm of both sides, we find $\kappa_2=\sqrt{\Lambda}$, where Λ is given in (8) and E_3 is given in (14). Notice that κ_2 is also a constant. Thus, γ is a θ_i-slant helix that belongs to (d) in the list.

Let us consider the Legendre case separately. In this case, since $\cos\theta_i=0, \forall i$, we have

$$\sum_{i=1}^{s}\cos^2\theta_i=0,\ \sum_{i=1}^{s}\alpha_i\cos\theta_i=0,\ \sum_{i=1}^{s}\cos\theta_i\xi_i=0,$$

which gives us

$$\kappa_1=|q|,\ \kappa_2=\sqrt{\sum_{i=1}^{s}\alpha_i^2}$$

using (10) and calculating $\Lambda=\sum_{i=1}^{s}\alpha_i^2$. Thus, γ is a Legendre helix that belongs to (c) in the list. One can easily see that $E_3\in sp\{T,\xi_1,\xi_2,\dots,\xi_s\}$ and the coefficients are constants. If we write

$$E_3=c_0T+\sum_{i=1}^{s}c_i\xi_i$$

for some constants $c_0,\dots,c_s$, we obtain

$$\begin{aligned}
\nabla_T E_3 &= -\kappa_2E_2+\kappa_3E_4 \\
&= c_0\nabla_T T+\sum_{i=1}^{s}c_i\nabla_T\xi_i \\
&= c_0\kappa_1E_2-\left(\sum_{i=1}^{s}c_i\alpha_i\right)fT.
\end{aligned}$$

For a normal magnetic curve, since $fT\parallel E_2$, we deduce that $\kappa_3=0$. So, the list is complete. □

The proof of this theorem also leads to a remarkable result that bounds the osculating order, making its inclusion here both meaningful and well-placed.

Corollary 1. *The osculating order of a normal magnetic curve in a homothetic s-th Sasakian manifold is at most* 3.

Proof. From the previous proof, The Gram–Schmidt process definitively concludes after we differentiate E_3 and find $\kappa_3 = 0$, if it has not already. If it concludes earlier, it would imply $r < 3$. In either case, r cannot exceed 3. □

In the next proposition, we present a nice result for Legendre helices in homothetic s-th Sasakian manifolds:

Proposition 2. *Let γ be a unit-speed Legendre helix of order 3 with $fT \parallel E_2$ in a homothetic s-th Sasakian manifold $(M^{2n+s}, f, \xi_i, \eta_i, g)$. Then, we have*

$$\kappa_2 = \sqrt{\sum_{i=1}^{s} \alpha_i^2},\ E_2 = \pm fT,\ E_3 = \frac{\pm 1}{\sqrt{\sum_{i=1}^{s} \alpha_i^2}} \sum_{i=1}^{s} \alpha_i \xi_i.$$

Proof. Since γ is a Legendre helix, we have $\cos\theta_i = 0$, $i = 1, 2, \ldots, s$, and κ_1, κ_2 are constants. $fT \parallel E_2$ gives us

$$fT = \lambda E_2$$

for some real valued differentiable function λ. Taking the norm of both sides, we have

$$\sqrt{g(fT, fT)} = |\lambda| \|E_2\|,$$

which is calculated as

$$\sqrt{g(T, T) - \sum_{i=1}^{s} \cos^2\theta_i} = |\lambda|.$$

So, we obtain $\lambda = \pm 1$ and

$$E_2 = \pm fT. \tag{15}$$

If we differentiate Equation (15), we find

$$\begin{aligned} -\kappa_1 T + \kappa_2 E_3 &= \pm \nabla_T fT \\ &= \pm \left(\sum_{i=1}^{s} \alpha_i \xi_i + \kappa_1 f E_2 \right). \end{aligned} \tag{16}$$

From Equation (15), if we apply f, we obtain

$$fE_2 = \pm f^2 T = \pm \left(-T + \sum_{i=1}^{s} \cos\theta_i \xi_i \right) = \mp T. \tag{17}$$

Equations (16) and (17) give us

$$\begin{aligned} -\kappa_1 T + \kappa_2 E_3 &= \pm \left(\sum_{i=1}^{s} \alpha_i \xi_i \mp \kappa_1 T \right) \\ &= \pm \sum_{i=1}^{s} \alpha_i \xi_i - \kappa_1 T, \end{aligned}$$

that is,

$$\kappa_2 E_3 = \pm \sum_{i=1}^{s} \alpha_i \xi_i.$$

The norm of this last equation concludes

$$\kappa_2 = \sqrt{\sum_{i=1}^{s} \alpha_i^2}.$$

Then, E_3 becomes

$$E_3 = \frac{\pm 1}{\sqrt{\sum\limits_{i=1}^{s} \alpha_i^2}} \sum_{i=1}^{s} \alpha_i \xi_i.$$

□

With the following theorem, we provide the criteria for the contact angles and the strength of the magnetic field that determine when θ_i-slant helices with $fT \parallel E_2$ will be normal magnetic curves:

Theorem 2. *Let γ be a unit-speed θ_i-slant helix of order $r \leq 3$ satisfying $fT \parallel E_2$, with given curvatures κ_1, κ_2 and contact angles θ_i $(i = 1, 2, \ldots, s)$ in a homothetic s-th Sasakian manifold $(M^{2n+s}, f, \xi_i, \eta_i, g)$. Then,*

(i) *If $\sum\limits_{i=1}^{s} \cos^2\theta_i = 1$, then γ is a geodesic as an integral curve of $\sum\limits_{i=1}^{s} \cos\theta_i \xi_i$; therefore, it is a normal magnetic curve for F_q with any q.*

(ii) *If $\sum\limits_{i=1}^{s} \cos^2\theta_i = 0$ and $\kappa_1 \neq 0$ (namely, γ is a non-geodesic Legendre curve), then γ is a normal magnetic curve for $F_{\mp\kappa_1}$.*

(iii) *If $\cos\theta_i = \frac{\varepsilon\alpha_i}{\sqrt{\kappa_1^2 + \sum\limits_{i=1}^{s}\alpha_i^2}}$, $(i = 1, 2, \ldots, s)$, then γ is a normal magnetic curve for $F_{\varepsilon\sqrt{\kappa_1^2 + \sum\limits_{i=1}^{s}\alpha_i^2}}$, where $\varepsilon = -sgn(g(fT, E_2))$. Given this situation, γ is a θ_i-slant circle.*

(iv) *If*

$$\sum_{i=1}^{s} \cos^2\theta_i = 1 - \frac{\kappa_1^2}{\Psi^2}, \tag{18}$$

then γ is a normal magnetic curve for $F_{\varepsilon\Psi}$, where we denote $\varepsilon = -sgn(g(fT, E_2))$ and

$$\Psi = \sqrt{\begin{array}{c}\left(\kappa_1 + \varepsilon\frac{\sum\limits_{i=1}^{s}\alpha_i\cos\theta_i}{\sqrt{1-\sum\limits_{i=1}^{s}\cos^2\theta_i}}\right)^2 + \kappa_2^2 + \left(\sum\limits_{i=1}^{s}\cos^2\theta_i - 2\right)\frac{\left(\sum\limits_{i=1}^{s}\alpha_i\cos\theta_i\right)^2}{1-\sum\limits_{i=1}^{s}\cos^2\theta_i} \\ + \left(\sum\limits_{i=1}^{s}\cos^2\theta_i - 1\right)\sum\limits_{i=1}^{s}\alpha_i^2\end{array}}. \tag{19}$$

(v) *If none of the above is satisfied, γ is not qualified as a normal magnetic curve for any F_q.*

Proof. Since γ is a unit-speed θ_i-slant helix, it is given that $\cos\theta_i$ $(i = 1, 2, \ldots, s)$ and κ_1, κ_2 are all constants. Furthermore, from $fT \parallel E_2$, we can write

$$fT = \lambda E_2 \tag{20}$$

for some differentiable function λ. Equation (20) gives us

$$g(fT, fT) = \lambda^2 g(E_2, E_2),$$

which is equivalent to

$$\lambda = \pm\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i}.$$

As a result, we obtain

$$fT = -\varepsilon\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} E_2, \tag{21}$$

where we denote $\varepsilon = -sgn(g(fT, E_2))$.

(i) If $\sum_{i=1}^{s} \cos^2\theta_i = 1$, then Equation (21) becomes

$$fT = 0.$$

Applying f, we find

$$f^2T = -T + \sum_{i=1}^{s} \cos\theta_i \xi_i = 0,$$

that is, $T = \sum_{i=1}^{s} \cos\theta_i \xi_i$. Then, we calculate

$$\begin{aligned} \nabla_T T &= \nabla_{\left(\sum_{i=1}^{s} \cos\theta_i \xi_i\right)} \left(\sum_{k=1}^{s} \cos\theta_k \xi_k\right) \\ &= \sum_{i,k=1}^{s} \cos\theta_i \cos\theta_k \nabla_{\xi_i} \xi_k = 0 \\ &= \kappa_1 E_2. \end{aligned}$$

Hence, we obtain $\kappa_1 = 0$, i.e., γ is a geodesic. We also have $\nabla_T T = 0 = -qfT$ for any q. Thus, γ is a normal magnetic curve for F_q with any q.

(ii) If $\sum_{i=1}^{s} \cos^2\theta_i = 0$ and $\kappa_1 \neq 0$, then $\cos\theta_i = 0$, $(i = 1, 2, \ldots, s)$; that is, γ is a Legendre curve. From Proposition 2, we have $E_2 = \pm fT$. Using Frenet equations, we calculate

$$\nabla_T T = \kappa_1 E_2 = \pm\kappa_1 fT.$$

As a result, γ becomes a normal magnetic curve for $F_{\mp\kappa_1}$.

(iii) Let

$$\cos\theta_i = \frac{\varepsilon\alpha_i}{\sqrt{\kappa_1^2 + \sum_{i=1}^{s} \alpha_i^2}}, (i = 1, 2, \ldots, s).$$

So, we find

$$\sum_{i=1}^{s} \cos^2\theta_i = \frac{\sum_{i=1}^{s} \alpha_i^2}{\kappa_1^2 + \sum_{i=1}^{s} \alpha_i^2},$$

$$1 - \sum_{i=1}^{s} \cos^2\theta_i = \frac{\kappa_1^2}{\kappa_1^2 + \sum_{i=1}^{s} \alpha_i^2},$$

and

$$\sqrt{1 - \sum_{i=1}^{s} \cos^2\theta_i} = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \sum_{i=1}^{s} \alpha_i^2}}.$$

From Equation (21), we can write

$$fT = \frac{-\varepsilon\kappa_1}{\sqrt{\kappa_1^2 + \sum\limits_{i=1}^{s}\alpha_i^2}}E_2.$$

Then, it is easy to see that

$$\begin{aligned} \nabla_T T &= \kappa_1 E_2 \\ &= -\varepsilon\sqrt{\kappa_1^2 + \sum_{i=1}^{s}\alpha_i^2} fT. \end{aligned}$$

As a result, γ becomes a normal magnetic curve for $F_{\varepsilon\sqrt{\kappa_1^2+\sum\limits_{i=1}^{s}\alpha_i^2}}$. Additionally, if we use Equation (8) for $q = \varepsilon\sqrt{\kappa_1^2 + \sum\limits_{i=1}^{s}\alpha_i^2}$, we find $\kappa_2 = \sqrt{\Lambda} = 0$. So, γ is a θ_i-slant circle.

(iv) Finally, let

$$\sum_{i=1}^{s}\cos^2\theta_i = 1 - \frac{\kappa_1^2}{\Psi^2},$$

where Ψ is as given in (19). From Equation (21), we have

$$\begin{aligned} fT &= -\varepsilon\sqrt{1 - \left(1 - \frac{\kappa_1^2}{\Psi^2}\right)}E_2 \\ &= -\varepsilon\frac{\kappa_1}{\Psi}E_2. \end{aligned}$$

Using the Frenet equations, we can write

$$\begin{aligned} \nabla_T T &= \kappa_1 E_2 = \kappa_1\left(-\varepsilon\frac{\Psi}{\kappa_1}\right)fT \\ &= -\varepsilon\Psi fT. \end{aligned}$$

As a result, γ becomes a normal magnetic curve for $F_{\varepsilon\Psi}$. Now, let us see how Ψ is calculated. Here, our aim is to determine $q = \varepsilon\Psi$ in terms of κ_1 and κ_2. We have already shown that γ is a normal magnetic curve for $F_{\varepsilon\Psi}$. So, we can use Theorem 1 (d). We can write

$$\kappa_1 = |q|\sqrt{1 - \sum_{i=1}^{s}\cos^2\theta_i}, \tag{22}$$

$$\kappa_2 = \sqrt{\Lambda}. \tag{23}$$

Equation (22) can be rewritten as

$$\left(\sum_{i=1}^{s}\cos^2\theta_i\right)q^2 = q^2 - \kappa_1^2. \tag{24}$$

Then, Equation (23) gives us

$$\begin{aligned}\kappa_2^2 &= \Lambda = \left(\sum_{i=1}^{s}\cos^2\theta_i\right)q^2 - 2\left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)q + \left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)^2 \\ &\quad + \left(1-\sum_{i=1}^{s}\cos^2\theta_i\right)\left(\sum_{i=1}^{s}\alpha_i^2\right), \\ &= \left(q^2-\kappa_1^2\right) - 2\left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)q + \left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)^2 \\ &\quad + \left(1-\sum_{i=1}^{s}\cos^2\theta_i\right)\left(\sum_{i=1}^{s}\alpha_i^2\right).\end{aligned} \tag{25}$$

Now, we assign

$$\sum_{i=1}^{s}\alpha_i\cos\theta_i = \sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}D$$

for some constant D. From Equation (22), we also know that

$$\varepsilon\kappa_1 = q\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i},$$

where $\varepsilon = -sgn(g(fT, E_2))$. We rearrange the terms in (25) as

$$\begin{aligned}-2\left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)q &= -2\sqrt{1-\sum_{i=1}^{s}\cos^2\theta_i}Dq, \\ &= -2\varepsilon\kappa_1 D\end{aligned} \tag{26}$$

and

$$\left(\sum_{i=1}^{s}\alpha_i\cos\theta_i\right)^2 = \left(1-\sum_{i=1}^{s}\cos^2\theta_i\right)D^2. \tag{27}$$

Finally, we write (26) and (27) in (25). Then, after completing the square to obtain $(\kappa_1+\varepsilon D)^2$, we leave the first term q^2 on the right-hand side of (25). As a result, we find Ψ as given in (19).

Since the list in Theorem 1 includes all cases where γ is a normal magnetic curve, then there does not exist any other normal magnetic curve in M, as stated in (v). □

4. Parameterization of Magnetic Curves in $\mathbb{R}^{2n+s}$ as a Homothetic s-th Sasakian Manifold

$\mathbb{R}^{2n+s}(-3s)$ is a well-known S-manifold [8], which is a specific kind of trans-S-manifold with $\alpha_i = 1, \beta_i = 0, i = 1, 2, \ldots, s$ [9]. Using generalized D-homothetic transformations, from Theorem 4.4 of [9], we can produce the following homothetic s-th Sasakian manifold using the structures of $\mathbb{R}^{2n+s}(-3s)$. The newly generated manifold is denoted in short by $M = (\mathbb{R}^{2n+s}, f, \xi_i, \eta_i, g)$.

Let us consider $\mathbb{R}^{2n+s}$ and its coordinate functions $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s\}$. Let a and b be positive real numbers. One can define

$$\xi_i = \frac{2}{a}\frac{\partial}{\partial z_i},\ i = 1, \ldots, s,$$

$$\eta_i = \frac{a}{2}\left(dz_i - \sum_{j=1}^{n} y_j dx_j\right),\ i = 1, \ldots, s,$$

$$fU = \sum_{j=1}^{n} V_j \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} U_j \frac{\partial}{\partial y_j} + \left(\sum_{j=1}^{n} V_j y_j \right) \left(\sum_{i=1}^{s} \frac{\partial}{\partial z_i} \right),$$

$$g = \sum_{i=1}^{s} \eta_i \otimes \eta_i + \frac{b}{4} \sum_{j=1}^{n} \left(dx_j \otimes dx_j + dy_j \otimes dy_j \right),$$

where

$$U = \sum_{j=1}^{n} \left(U_j \frac{\partial}{\partial x_j} + V_j \frac{\partial}{\partial y_j} \right) + \sum_{i=1}^{s} \left(W_i \frac{\partial}{\partial z_i} \right) \in \chi(M).$$

Let us denote $M = (\mathbb{R}^{2n+s}, f, \xi_i, \eta_i, g)$. Using Theorem 4.4 of [9], M becomes a trans-S-manifold with

$$\alpha_i = \frac{a}{b},\ \beta_i = 0,\ i = 1, 2, \ldots, s,$$

that is, M is a homothetic s-th Sasakian manifold. The vector fields

$$U_j = \frac{2}{\sqrt{b}} \frac{\partial}{\partial y_j},\ U_{n+j} = fU_j = \frac{2}{\sqrt{b}} \left(\frac{\partial}{\partial x_j} + y_j \sum_{i=1}^{s} \frac{\partial}{\partial z_i} \right),\ \xi_i = \frac{2}{a} \frac{\partial}{\partial z_i}$$

are g-orthonormal. The Riemannian connection associated to g can be calculated directly from equation (4.16) of [9] as

$$\nabla_{U_j} U_k = \nabla_{U_{n+j}} U_{n+k} = 0,\ \nabla_{U_j} U_{n+k} = \frac{a}{b} \delta_{jk} \sum_{i=1}^{s} \xi_i,\ \nabla_{U_{n+j}} U_k = \frac{-a}{b} \delta_{jk} \sum_{i=1}^{s} \xi_i,$$

$$\nabla_{U_j} \xi_i = \nabla_{\xi_i} U_j = \frac{-a}{b} U_{n+j},\ \nabla_{U_{n+j}} \xi_i = \nabla_{\xi_i} U_{n+j} = \frac{a}{b} U_j.$$

Now, we can give the following theorem:

Theorem 3. *Normal magnetic curves on $M = (\mathbb{R}^{2n+s}, f, \xi_i, \eta_i, g)$ that satisfy the Lorentz equation $\nabla_T T = -qfT$ are described by the following parametric equations:*

(a)

$$\gamma_j(t) = \frac{c_j}{-\lambda} \sin f_j(t) + b_j, \tag{28}$$

$$\gamma_{n+j}(t) = \frac{c_j}{\lambda} \cos f_j(t) + d_j, \tag{29}$$

$$\gamma_{2n+i}(t) = \frac{2\cos\theta_i}{a} t - \sum_{j=1}^{n} \left\{ \frac{c_j^2}{4\lambda^2} \left[\sin\left(2f_j(t)\right) + 2f_j(t) \right] + \frac{c_j d_j}{\lambda} \sin f_j(t) \right\} + h_i, \tag{30}$$

$$f_j(t) = -\lambda t + a_j,$$

$$i = 1, \ldots, s,\ j = 1, 2, \ldots, n,$$

$$\lambda = -q + \frac{2a}{b} \left(\sum_{i=1}^{s} \cos\theta_i \right) \neq 0,$$

where a_j, b_j, c_j, d_j, and h_i are arbitrary constants such that c_j satisfies

$$\sum_{j=1}^{n} c_j^2 = \frac{4}{b} \left(1 - \sum_{i=1}^{s} \cos^2\theta_i \right); \tag{31}$$

or

(b)

$$\gamma_j(t) = c_j t + d_j,$$

$$\gamma_{n+j}(t) = c_{n+j} t + d_{n+j},$$

$$\gamma_{2n+i}(t) = \frac{2\cos\theta_i}{a}t + \sum_{j=1}^{n} c_j\left(\frac{c_{n+j}}{2}t^2 + d_{n+j}t\right) + h_i,$$

$$i = 1,\ldots,s,\ j = 1,\ldots,n,$$

$$q = \frac{2a}{b}\left(\sum_{i=1}^{s}\cos\theta_i\right),$$

where c_j, c_{n+j}, d_j, d_{n+j}, and h_i are arbitrary constants such that c_j and c_{n+j} satisfy

$$\sum_{j=1}^{n}\left(c_j^2 + c_{n+j}^2\right) = \frac{4}{b}\left(1 - \sum_{i=1}^{s}\cos^2\theta_i\right).$$

Proof. Let $\gamma : I \to M$ be a normal magnetic curve. Then, Proposition 1 gives us that γ is a θ_i-slant curve. Moreover, from Corollary 1, its osculating order is at most 3. Let

$$\gamma(t) = (\gamma_1(t),\ldots,\gamma_n(t),\gamma_{n+1}(t),\ldots,\gamma_{2n}(t),\gamma_{2n+1}(t),\ldots,\gamma_{2n+s}(t))$$

denote the parameterization of γ, where t is the arc-length parameter. As a result, its tangential vector field T becomes

$$T = \sum_{j=1}^{n}\gamma_j'\frac{\partial}{\partial x_j} + \sum_{j=1}^{n}\gamma_{n+j}'\frac{\partial}{\partial y_j} + \sum_{i=1}^{s}\gamma_{2n+i}'\frac{\partial}{\partial z_i}.$$

It is more useful to write T in terms of the g-orthonormal basis as

$$\begin{aligned} T \quad = \quad & \frac{\sqrt{b}}{2}\left(\sum_{j=1}^{n}\gamma_{n+j}'U_j + \sum_{j=1}^{n}\gamma_j'U_{n+j}\right) \\ & + \frac{a}{2}\left[\sum_{i=1}^{s}\left(\gamma_{2n+i}' - \sum_{j=1}^{n}\gamma_j'\gamma_{n+j}\right)\xi_i\right]. \end{aligned}$$

Using the fact that γ is a θ_i-slant curve, we obtain

$$\eta_i(T) = \cos\theta_i,\ i = 1,2,\ldots,s,$$

which is equivalent to

$$\gamma_{2n+i}' = \frac{2\cos\theta_i}{a} + \sum_{j=1}^{n}\gamma_j'\gamma_{n+j}. \tag{32}$$

We also have $g(T,T) = 1$, so one can easily calculate

$$\sum_{j=1}^{n}\left(\gamma_j'\right)^2 + \sum_{j=1}^{n}\left(\gamma_{n+j}'\right)^2 = \frac{4}{b}\left(1 - \sum_{i=1}^{s}\cos^2\theta_i\right). \tag{33}$$

So now, we need the Lorentz equation

$$\nabla_T T = -qfT \tag{34}$$

to be satisfied by γ, since it is a normal magnetic curve. $\nabla_T T$ and fT can be calculated as

$$\nabla_T T = \frac{\sqrt{b}}{2}\left\{\begin{array}{c} \sum\limits_{j=1}^{n}\left[\gamma_{n+j}'' + \frac{2a}{b}\left(\sum\limits_{i=1}^{s}\cos\theta_i\right)\gamma_j'\right]U_j \\ +\sum\limits_{j=1}^{n}\left[\gamma_j'' - \frac{2a}{b}\left(\sum\limits_{i=1}^{s}\cos\theta_i\right)\gamma_{n+j}'\right]U_{n+j} \end{array}\right\}, \tag{35}$$

and

$$fT = \frac{\sqrt{b}}{2}\left[\sum_{j=1}^{n}\left(-\gamma_j'\right)U_j + \sum_{j=1}^{n}\gamma_{n+j}'U_{n+j}\right]. \quad (36)$$

From Equation (34), we deduce that $\nabla_T T \parallel fT$, i.e., the corresponding coefficients of their unique representations in terms of the g-orthonormal basis must be proportional, and it is easy to see that the proportionality constant is $(-q)$. Simplifying by canceling $\sqrt{b}/2$, we obtain

$$\frac{\gamma_{n+j}'' + \frac{2a}{b}\left(\sum_{i=1}^{s}\cos\theta_i\right)\gamma_j'}{-\gamma_j'} = \frac{\gamma_j'' - \frac{2a}{b}\left(\sum_{i=1}^{s}\cos\theta_i\right)\gamma_{n+j}'}{\gamma_{n+j}'} = -q,\ j = 1,2,\ldots,n,$$

which can be rewritten as

$$\frac{\gamma_{n+j}''}{-\gamma_j'} = \frac{\gamma_j''}{\gamma_{n+j}'} = \lambda,\ j = 1,2,\ldots,n, \quad (37)$$

where we denote the corresponding proportionality constant by λ as

$$\lambda = -q + \frac{2a}{b}\left(\sum_{i=1}^{s}\cos\theta_i\right).$$

Firstly, let $\lambda \neq 0$. Then we have the following ODEs from (37) for all $j = 1,2,\ldots,n$:

$$\gamma_{n+j}''\gamma_{n+j}' + \gamma_j''\gamma_j' = 0,$$

which are integrated to find

$$\left(\gamma_j'\right)^2 + \left(\gamma_{n+j}'\right)^2 = c_j^2,$$

for some arbitrary constant c_j. These circular equations are to be solved by using

$$\gamma_j' = c_j\cos f_j,\ \gamma_{n+j}' = c_j\sin f_j, \quad (38)$$

where $f_j : I \to \mathbb{R}$, $(j = 1,2,\ldots,s)$ are functions of the arc-length parameter t. From Equations (37) and (38), we find

$$f_j' = -\lambda,$$

that is,

$$f_j(t) = -\lambda t + a_i,$$

for some arbitrary constant a_i. If we replace f_j in (38) and integrate, we obtain (28) and (29). Then, we use these in (32) and find (30). Equation (31) is calculated by using the fact that γ is unit-speed, i.e., $g(T,T) = 1$. The proof of (a) is now complete.

Finally, let $\lambda = 0$. Following the same procedure, we obtain the linear equations of γ_j, γ_{n+j}, and the parabolic equations of γ_{2n+i} as given in (b). In fact, from Equation (37) and $\lambda = 0$, we obtain

$$\gamma_j'' = \gamma_{n+j}'' = 0,$$

from which the proof of (b) is straightforward. □

We conclude our study with two explicit examples:

Example 1. *Let $n = 1$, $s = 2$, $a = 2$ and $b = 4$. Then $\gamma : I \to M$,*

$$\gamma(t) = \left(\frac{1}{2}t, \frac{1}{2}t, \frac{1}{8}t^2 + \frac{1}{2}t, \frac{1}{8}t^2 + \frac{\sqrt{2}}{2}t \right)$$

is a normal magnetic curve for $F_{\frac{1+\sqrt{2}}{2}}$. It is a θ_i-slant curve with contact angles $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{\pi}{4}$. It satisfies Theorem 3 (b).

Example 2. *Let $n = 1$, $s = 3$, $a = 3$ and $b = 2$. Then $\gamma : I \to M$,*

$$\gamma(t) = \left(\sin t, -\cos t, \frac{-1}{6}t - \frac{1}{4}\sin 2t, \frac{-5}{6}t - \frac{1}{4}\sin 2t, \frac{-1}{2}t - \frac{1}{4}\sin 2t \right)$$

is a normal magnetic curve for F_1. It is a θ_i-slant curve with contact angles $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{2\pi}{3}$, and $\theta_3 = \frac{\pi}{2}$. It satisfies Theorem 3 (a).

5. Discussion

If we select $\alpha_i = 1$, $i = 1, 2, \ldots, s$, then a homothetic s-th Sasakian manifold becomes an S-manifold. Therefore, our new results not only complete our previous study [24] for normal magnetic curves in S-manifolds when the contact angles do not necessarily need to be equal but also generalize those prior results to a broader class of manifolds. Our parameterization Theorem 3 can be considered in a similar manner, since $a = b = 1$ gives $M = \mathbb{R}^{2n+s}(-3s)$, and then γ represents all of the normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$ without any requirement on the contact angles.

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