EXTENDED HECKE GROUPS $\overline{H}(\lambda_q)$ AND THEIR FUNDAMENTAL REGIONS[1]

by
Recep Sahin*, **Sebahattin Ikikardes*** & **Ozden Koruoğlu****

*Balıkesir Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 10145 Çağış Kampüsü, Balıkesir, Turkey
rsahin@balikesir.edu.tr, skardes@balikesir.edu.tr
&
**Balıkesir Üniversitesi, Necatibey Eğitim Fakültesi, İlköğretim Bölümü, Matematik Eğitimi, 10100 Balıkesir, Turkey
ozdenk@balikesir.edu.tr

Abstract

We consider the extended Hecke groups $\overline{H}(\lambda_q)$ generated by $T(z) = -1/\ z$, $S(z) = -1/(z + \lambda_q)$ and $R(z) = 1/\overline{z}$ with $\lambda_q = 2\cos(\pi/q)$ for $q \geq 3$ an integer. In this work, we give abstract group structure and fundamental region of the extended Hecke groups $\overline{H}(\lambda_q)$.

1. Introduction

In [3], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

E. Hecke showed that H(λ) is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, where $q \geq 3, q \in \mathbb{N}$ or $\lambda \geq 2$. In these two cases $H(\lambda)$ is called a *Hecke group*. We consider the former case. Then the Hecke group $H(\lambda_q)$ is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by T and S, and it has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q, \ [1].$$

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The most important and worked Hecke group is the modular group $H(\lambda_3)$. In this case $\lambda_3=2\cos\frac{\pi}{3}=1$, i.e. all coefficients of the elements of $H(\lambda_3)$ are rational integers. The next two most important Hecke groups are those for $q=4$ and 6, in which cases $\lambda_q=\sqrt{2}$ and $\sqrt{3}$, respectively.

The next most interesting q is 5 because it is the only other value of q for which $\mathbb{Q}(\lambda_q)$ is a quadratic field, and also because of Leutbecher's result which essentially states that for $q=5$ all elements of the field $\mathbb{Q}(\lambda_5)$ are cusp points. The elements of $H(\lambda_5)$ are worked out by D. Rosen in [9].

The extended modular group $\overline{H}(\lambda_3)$ has been defined in (see [5], [6], [17]) by adding the reflection $R(z)=1/\overline{z}$ to the generators of the modular group $H(\lambda_3)$. Also, the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [8], [10] and [11], similar to the extended modular group by adding the reflection $R(z)=1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$ where $q\geq 3$ integer. Some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ (commutator subgroups, even subgroups, principal congruence subgroups, Fuchsian subgroups) and certain relations between them had been studied in [8], [10] and [11]. Additionally, the power and free subgroups of the extended modular group $\overline{H}(\lambda_3)$, the extended Hecke group $\overline{H}(\lambda_5)$ and the extended Hecke groups $\overline{H}(\lambda_q)$, $q\geq 3$ prime number, and the relations between power subgroups and commutator subgroups and other subgroups of finite index (especially, index 2, 4, $2q$) have been investigated by the authors in [12], [13], and [15].

In this work we give a new proof of the fact that the extended Hecke groups $\overline{H}(\lambda_q)$ is isomorphic to the free product of two finite dihedral groups of orders 4 and $2q$ with amalgamation $\mathbb{Z}_2$. Also we find their fundamental region.

2. Decomposition of the Extended Hecke Groups $\overline{H}(\lambda_q)$.

It is known, [8], [10] and [11], that the extended Hecke group $\overline{H}(\lambda_q)$ has a presentation

$$\overline{H}(\lambda_q) =< R_1,R_2,R_3 \mid R_1^2=R_2^2=R_3^2=(R_1R_2)^q=(R_3R_1)^2=I \ >$$

where

$$R_1(z)=\frac{1}{\overline{z}}, \qquad R_2(z)=\frac{-\overline{z}}{\lambda_q\overline{z}+1}, \qquad R_3(z)=-\overline{z}.$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. It has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_3,$$

where

$$T = R_3R_1 = R_1R_3, \qquad S = R_1R_2.$$

Putting $R = R_1$, we have

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = I,\ RT = TR, RS = S^{-1}R >$$

or

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (SR)^2 = I > . \qquad (2.1)$$

The signature of $\overline{H}(\lambda_q)$ is $(0; +; [-]; \{(2, q, \infty)\})$ and the quotient $U^*/\overline{H}(\lambda_q)$, where $U^* = U \cup \mathbb{Q} \cup \{\infty\}$ and U is the upper half-plane, is a disc whose boundary contains two branch points and one cusp. Since the extended Hecke groups $\overline{H}(\lambda_q)$ contain a reflection, they are NEC groups. Thus quotient space $U^*/\overline{H}(\lambda_q)$ is a Klein surface. Also $U^*/H(\lambda_q)$ is the canonical double cover of $U^*/\overline{H}(\lambda_q)$.

Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$ and it has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q,$$

where

$$T = R_3R_1 = R_1R_3, \qquad S = R_1R_2.$$

The signature of $H(\lambda_q)$ is $(0; +; [2, q, \infty]; \{(-)\})$ and the quotient $U/H(\lambda_q)$ is a sphere with one puncture, two elliptic fixed points of orders 2 and q. Also it is easy see that the quotient space $U^*/H(\lambda_q)$ is a Riemann surface.

Now we can write the following theorem about abstract group structure of the extended Hecke groups $\overline{H}(\lambda_q)$:

Theorem 2.1. *The extended Hecke group $\overline{H}(\lambda_q)$ is given directly as a free product of two groups G_1, G_2 with amalgamated subgroup $\mathbb{Z}_2$, where G_1 is the dihedral group D_2 and G_2 is the dihedral group D_q, that is $\overline{H}(\lambda_q) \cong D_2 *_{\mathbb{Z}_2} D_q$.*

Proof. The result follows from a presentation of the extended Hecke group $\overline{H}(\lambda_q)$ given (2.1):

$$\overline{H}(\lambda_q) =< T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = I > .$$

Let $G_1 =< T, R \mid T^2 = R^2 = (TR)^2 = I >\cong D_2$, and let $G_2 =< S, R \mid S^q = R^2 = (SR)^2 = I >\cong D_q$. Then $\overline{H}(\lambda_q)$ is $G_1 * G_2$ with the identification $R = R$.

In G_1, the subgroup generated by R is $\mathbb{Z}_2$,this is also true in G_2. Therefore the identification induces an isomorphism and $\overline{H}(\lambda_q)$ is a generalized free product with the subgroup $M \cong \mathbb{Z}_2$ amalgamated. $\square$

Notice that this result coincides with the ones given in [4] for the generalized Hecke groups.

As is well-known, fundamental region plays an important role in the geometrical study of a group and its subgroups. Therefore, we shall have a great deal of interest in their fundamental regions.

3. Fundamental Region of $\overline{H}(\lambda_q)$

Let us begin with the modular group $H(\lambda_3)$. A fundamental region for $H(\lambda_3)$ is given by

$$F = \{z \in U : |z| > 1, \ \ |Re\ z| < 1/2\}.$$

where U denotes the upper-half plane [16, pp.34].

E.Hecke gave a generalization of F to all Hecke groups $H(\lambda_q)$: He showed that, when $\lambda \geq 2$ and real, or when $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, the set

$$F_\lambda = \{z \in U : |Re\ z| < \lambda/2, \ \ |z| > 1\}$$

is a fundamental region for the group $H(\lambda)$, and also F_λ fails to be a fundamental region for all other $\lambda > 0$, [3].

We therefore take a fundamental region for $H(\lambda_q)$ as

$$F_{\lambda_q} = \{z \in U : |Re\ z| < \lambda_q/2, \ \ |z| > 1\}.$$

It is well-known that fundamental region of a group is not unique. We have already seen that $F_{\lambda_q} = F_1 \cup F_2$ in Figure 1 is a fundamental region for $H(\lambda_q)$. Actually a shaded region together with an unshaded one form a fundamental region for $H(\lambda_q)$. Therefore sometimes, for convenience, we shall take it as

$$F'_{\lambda_q} = \{z \in U : -\frac{\lambda_q}{2} < Re\ z < 0, \ \ \left|z + \frac{1}{\lambda_q}\right| > \frac{1}{\lambda_q}\}.$$

which is $F_1 \cup TF_2$. The elliptic generator S has order q so that the q transforms of $F'_{\lambda_q} = F_1 \cup TF_2$ form a pattern around the center point $-\overline{\zeta} = -e^{-i\pi/q}$ which is the fixed point of S. In another words the transforms of F_1 and TF_2 under S form a pattern alternately. The region F_1 has vertices i, ζ and ∞ in the upper half plane U.

Also, by the Riemann-Hurwitz formula, the hyperbolic area $\mu(H(\lambda_q))$ of the fundamental region of $H(\lambda_q)$ is $2\pi(1 - 1/2 - 1/q) = \pi(q-2)/q$.

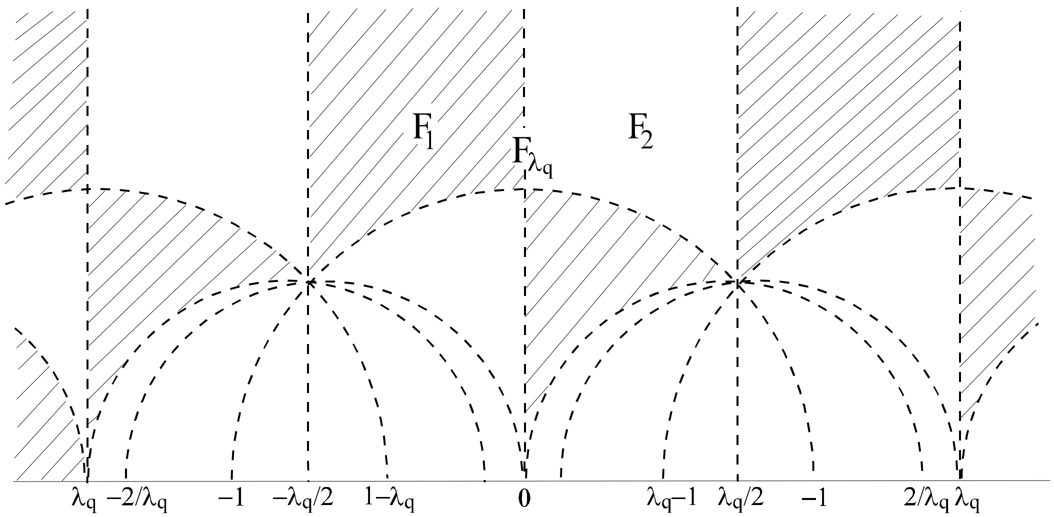

Figure 1

Then, Schoeneberg showed that the set

$$\overline{F} = \{z \in U : |z| > 1, \ -1/2 < Re\ z < 0\}.$$

is a fundamental region for the group $\overline{H}(\lambda_3)$, [16, pp.34].

Now, we can find a fundamental region of extended Hecke groups $\overline{H}(\lambda_q)$ using the above results.

Theorem 3.1. *The set*

$$\overline{F}_{\lambda_q} = \{z \in U : |z| > 1, -\lambda_q/2 < Re\ z < 0\}$$

is a fundamental region of extended Hecke groups $\overline{H}(\lambda_q)$.

Proof. We know that the set

$$F_{\lambda_q} = \{z \in U : |Re\ z| < \lambda_q/2, \ |z| > 1\}$$

is a fundamental region of extended Hecke groups $\overline{H}(\lambda_q)$. The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in the extended Hecke group $\overline{H}(\lambda_q)$. If we use the Riemann-Hurwitz formula

$$\left[\overline{H}(\lambda_q) : H(\lambda_q)\right] = \frac{\mu(H(\lambda_q))}{\mu(\overline{H}(\lambda_q))}$$

where $\mu(H(\lambda_q))$ denotes the hyperbolic area of a fundamental region for $H(\lambda_q)$. Then we find the fundamental region of extended Hecke groups $\overline{H}(\lambda_q)$ as half of a fundamental region of Hecke groups $H(\lambda_q)$ in Figure 2, since every point in F_2 is equivalent to a point in F_1 under R_3, which is the reflection on the line $x = 0$, i.e. $R_3(F_2) = F_1$.

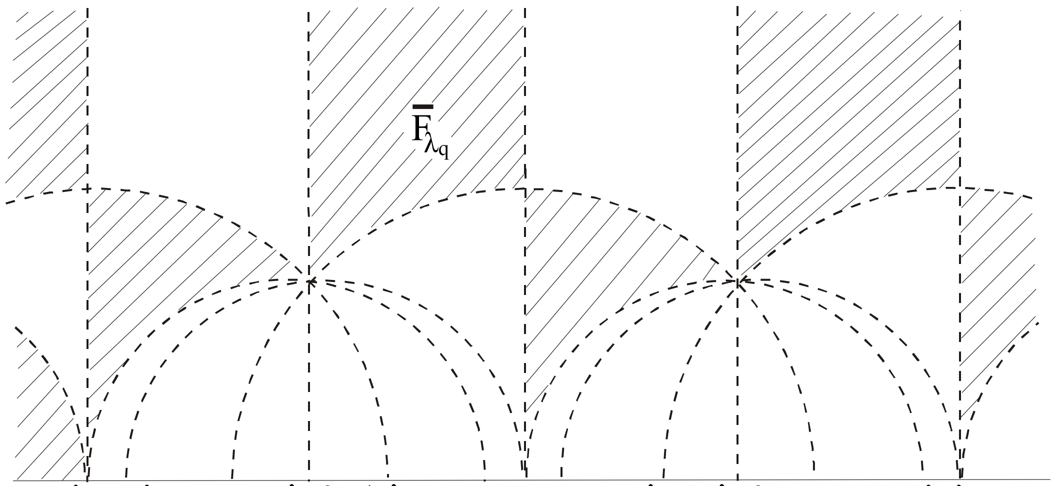

Figure 2

Let us now see $\overline{H}(\lambda_q)-$equivalency of any two points in $\overline{F}_{\lambda_q}$. We assume that any two points z_1, z_2 of $\overline{F}_{\lambda_q}$ be $\overline{H}(\lambda_q)-$equivalent. Then either $A(z_1) = z_2$ or $R_3A(z_1) = z_2$, i.e. $A(z_1) = -\,\overline{z}_2$, where $A \in H(\lambda_q)$.

i) If $A(z_1) = z_2$ then since $A \in H(\lambda_q)$ and $\overline{F}_{\lambda_q} \subset F_{\lambda_q}$, we have all following possibilities:

$$z_1 = z_2 = i \text{ and } A = T$$

$$z_1 = z_2 = \zeta \text{ and } A = S,\ ...,\ S^p,\ 1 \le p \le q-1$$

$$z_1 = z_2 = \infty \text{ and } A = U^n (n \in \mathbb{Z}).$$

ii) If $A(z_1) = -\,\overline{z}_2$ and $A \in H(\lambda_q)$ then we have all following possibilities:

$$Re\ (z_1) = 0 \text{ and } A = T \text{ so } R_3A = R_1$$

$$Re\ (z_1) = -\lambda_q/2 \text{ and } A = U \text{ so } R_3A = R_2$$

$$|z_1| = 1 \text{ and } A = I \text{ so } R_3A = R_3$$

$$z_1 = \zeta \text{ and } A = US,\ ...,\ US^p,\ 1 \le p \le q-1.$$

In all cases we find that $z_1 = z_2$. This shows that boundary points of $\overline{F}_{\lambda_q}$ can only be $\overline{H}(\lambda_q)-$equivalent to boundary points of $\overline{F}_{\lambda_q}$, that is in $\overline{F}_{\lambda_q}$ there are no $\overline{H}(\lambda_q)-$equivalent points. Thus the set $\overline{F}_{\lambda_q}$ is a fundamental region for $\overline{H}(\lambda_q)$. □

Corollary 3.2. *The hyperbolic area $\mu(\overline{H}(\lambda_q))$ of the fundamental region of $\overline{H}(\lambda_q)$ is $\pi(q-2)/2q$.*

Proof. As $\overline{H}(\lambda_q)$ has the signature $(0;+;[-];\{(2,q,\infty)\})$, and by the Riemann-Hurwitz formula, we get

$$\mu(\overline{H}(\lambda_q)) = 2\pi(-1+1/2(1/2+(q-1)/q+1) = \pi(q-2)/2q.$$

□

From the above result $\mu(H(\lambda_q)) = 2\ \mu(\overline{H}(\lambda_q))$. Thus the proof of the theorem 3.1 follows exactly the same way.

Also, in our recent paper [14], we showed that there is a relation between the extended Hecke groups $\overline{H}(\lambda_q)$ and the automorphism groups of compact bordered Klein surfaces of algebraic genus $p \geq 2$.

Now suppose that a bordered surface group Γ is a normal subgroup of finite index in $\overline{H}(\lambda_q)$. Then $\overline{H}(\lambda_q)/\Gamma$ is a group of automorphisms of the compact bordered Klein surface $X = U^*/\overline{H}(\lambda_q)$. If Γ is of genus $p \geq 2$, then

$$\left[\overline{H}(\lambda_q) : \Gamma\right] = \frac{2\pi(p-1)}{\pi(q-2)/2q} = \frac{4q}{(q-2)}(p-1).$$

Thus, finite quotient groups of all the extended Hecke groups $\overline{H}(\lambda_q)$, $q \geq 3$ integer, are the automorphism groups G that acts on compact bordered Klein surface X of genus $p \geq 2$, of order $\frac{4q}{(q-2)}(p-1)$. For example, the finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_3)$ (the extended modular group $PGL(2,\mathbb{Z})$), $\overline{H}(\lambda_4)$ or $\overline{H}(\lambda_5)$ are the groups of orders $|G| = 12(p-1)$, $|G| = 8(p-1)$, $|G| = \frac{20}{3}(p-1)$, respectively. Here the orders of these groups are the higgest three among the automorphism groups of surfaces of genus $p \geq 2$ (see [7, p.221, proposition 1]).

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