mathematics

MDPI

Article

New Fuzzy Topologies via Ideals and Generalized Openness

Ahu Açıkgöz

Department of Mathematics, Faculty of Science and Literature, Balıkesir University, 10145 Balıkesir, Turkey; ahuacikgoz@balikesir.edu.tr

Abstract

This paper introduces and investigates a new class of generalized open sets, called fuzzy $h_{\mathcal{I}}$-open sets, in fuzzy ideal topological spaces $(X, \tilde{\tau}, \tilde{\mathcal{I}})$. We prove that the collection of all fuzzy $h_{\mathcal{I}}$-open sets forms a fuzzy topology $\tilde{\tau}^{h_{\mathcal{I}}}$ satisfying $\tilde{\tau} \subseteq \tilde{\tau}^{h_{\mathcal{I}}}$ and show that $\tilde{\tau}^*$ and $\tilde{\tau}^{h_{\mathcal{I}}}$ are in general incomparable, demonstrating that the $h_{\mathcal{I}}$-construction captures fundamentally different information from the $*$-topology. We establish precise conditions under which these topologies coincide and introduce a fuzzy $h_{\mathcal{I}}$-T_1 separation axiom. Furthermore, we develop a comprehensive hierarchy of generalizations—fuzzy $h\alpha_{\mathcal{I}}$-open, fuzzy $hp_{\mathcal{I}}$-open, fuzzy $hs_{\mathcal{I}}$-open, and fuzzy $h\beta_{\mathcal{I}}$-open sets—and prove that these classes are pairwise distinct through genuinely fuzzy (non-characteristic) examples. We introduce fuzzy $h_{\mathcal{I}}$-continuous and fuzzy $h_{\mathcal{I}}$-irresolute functions, providing six equivalent characterizations and a closed-set criterion via the $*$-interior operator. The framework is applied to a concrete multi-criteria decision-making problem, where the ideal filters negligible criteria and the $h_{\mathcal{I}}$-interior provides a refined ranking that demonstrably outperforms the original fuzzy topology.

Keywords: fuzzy ideal topological space; h-open set; fuzzy $h_{\mathcal{I}}$-open set; fuzzy $h_{\mathcal{I}}$-interior; fuzzy $h_{\mathcal{I}}$-continuous function; fuzzy $h_{\mathcal{I}}$-irresolute function; generalized fuzzy topology; fuzzy $h_{\mathcal{I}}$-T_1 separation; multi-criteria decision making

MSC: 54A05; 03E72; 54C08; 90B50

1. Introduction

The theory of fuzzy sets, introduced by Zadeh [1] in his seminal 1965 paper, has fundamentally transformed how mathematicians and scientists model uncertainty, vagueness, and imprecision. Unlike classical set theory, where an element either belongs to a set or does not, fuzzy set theory allows for partial membership through membership functions taking values in the interval $[0, 1]$. This mathematical framework has proven invaluable for capturing the inherent fuzziness of real-world concepts and has found applications across diverse fields including control systems, pattern recognition, decision making, and artificial intelligence [2,3].

Chang [4] pioneered the extension of general topology to fuzzy sets by introducing fuzzy topological spaces, initiating a rich area of research that continues to attract significant attention. Subsequently, various generalizations of fuzzy open sets have been developed, including fuzzy semi-open sets by Azad [5], fuzzy pre-open sets and fuzzy α-open sets by Bin Shahna [6], and fuzzy β-open sets, creating a hierarchy of increasingly general openness concepts.

The practical relevance of fuzzy topology extends well beyond pure mathematics. In the context of image processing and computer vision, Rosenfeld [7] established the

Check for updates

Academic Editors: Hsien-Chung Wu, Ziye Zhang and Rongrong Yu

Received: 6 February 2026
Revised: 3 March 2026
Accepted: 5 March 2026
Published: 6 March 2026

 https://doi.org/10.3390/math14050904

foundations of fuzzy digital topology, demonstrating how fuzzy topological concepts such as connectedness, surroundedness, and boundary detection could be effectively applied to digital image analysis. More recently, Saha and Udupa [8] have shown that fuzzy digital topological and geometric methods play crucial roles in medical image processing, including in applications such as brain cortex segmentation, bone strength analysis, and vascular imaging, where noise, limited resolution, and background inhomogeneity lead to inherently fuzzy representations of target objects. Bloch [2] provided a comprehensive survey demonstrating that fuzzy sets and fuzzy topological structures are effectively employed in image segmentation, spatial reasoning, mathematical morphology, and multi-criteria decision making. In the geosciences, fuzzy methods have also proven effective for reservoir characterization; for instance, Moosavi et al. [9] recently demonstrated that combining fuzzy membership functions with support vector regression significantly improves the robustness of porosity estimation in the presence of noisy well-log data, illustrating the practical impact of fuzzy set theory in engineering applications.

The emergence of topological deep learning (TDL) represents a paradigm shift in machine learning, combining topological data analysis with deep learning methodologies [10,11]. This rapidly growing field has demonstrated that topological structures provide valuable insights for analyzing complex, high-dimensional data. The integration of fuzzy topological concepts into this framework offers promising avenues for handling uncertainty in data-driven applications. Furthermore, recent advances in rough set theory [12,13] have established strong connections between topological structures and data analysis, particularly in handling ambiguous and incomplete information systems.

The concept of ideals in topological spaces has deep roots in classical topology, dating back to the work of Kuratowski [14] and Vaidyanathaswamy [15]. Janković and Hamlett [16] significantly advanced this theory by introducing compatibility conditions between ideals and topologies. The extension of ideals to fuzzy settings by Sarkar [17] established the foundation for fuzzy ideal topological spaces, introducing fuzzy ideals and fuzzy local functions. The role of fuzzy ideals is particularly important: they provide a systematic mechanism for excluding negligible or irrelevant fuzzy sets from consideration. This mechanism has significant practical implications in applications such as noise filtering in image processing and outlier removal in data mining, where certain data components must be disregarded to obtain meaningful results [18,19].

Recently, Abbas [20] introduced a new class of open sets called h-open sets in topological spaces and studied h-continuous, h-irresolute, and h-totally continuous functions along with the associated operators. Subsequently, Acikgoz and Noiri [21] investigated the idealizability of h-open sets in ideal topological spaces.

In the present paper, we introduce a new class of generalized open sets called fuzzy $h_{\mathcal{I}}$-open sets in fuzzy ideal topological spaces $(X, \tilde{\tau}, \tilde{\mathcal{I}})$. This concept extends the h-open sets of Abbas [20] to the fuzzy ideal setting and provides a framework that incorporates ideal-based corrections to the fuzzy topology. Our main contributions are as follows:

(C1) We establish that $\tilde{\tau}^{h_{\mathcal{I}}}$ is a fuzzy topology satisfying $\tilde{\tau} \subseteq \tilde{\tau}^{h_{\mathcal{I}}}$ and prove that $\tilde{\tau}^*$ and $\tilde{\tau}^{h_{\mathcal{I}}}$ are in general incomparable (neither $\tilde{\tau}^* \subseteq \tilde{\tau}^{h_{\mathcal{I}}}$ nor $\tilde{\tau}^{h_{\mathcal{I}}} \subseteq \tilde{\tau}^*$ holds in general), characterizing when these topologies coincide. This shows that the $h_{\mathcal{I}}$-construction captures fundamentally different information from the $*$-topology.

(C2) We introduce a fuzzy $h_{\mathcal{I}}$-T_1 separation axiom and a subspace construction, demonstrating that $\tilde{\tau}^{h_{\mathcal{I}}}$ supports a rich topological theory beyond the generation of open sets.

(C3) We provide genuinely fuzzy (non-characteristic function) examples throughout the paper, illustrating that the theory captures nuanced interactions between membership degrees and ideal membership that have no crisp counterpart.

(C4) We present a concrete application to multi-criteria decision making, where the fuzzy ideal filters negligible criteria and the $h_{\mathcal{I}}$-interior produces a refined alternative ranking that demonstrably outperforms the standard fuzzy topological approach.

The paper is organized as follows: Section 2 presents the necessary preliminaries. Section 3 introduces fuzzy $h_{\mathcal{I}}$-open sets and establishes their fundamental properties, including the incomparability of $\tilde{\tau}^*$ and $\tilde{\tau}^{h_{\mathcal{I}}}$. Section 4 develops the hierarchy of generalizations with genuinely fuzzy examples. Section 5 introduces the fuzzy $h_{\mathcal{I}}$-interior and closure operators with detailed proofs. Section 6 introduces a fuzzy $h_{\mathcal{I}}$-T_1 separation axiom and fuzzy $h_{\mathcal{I}}$-subspaces. Section 7 studies fuzzy $h_{\mathcal{I}}$-continuous functions with six equivalent characterizations. Section 8 investigates fuzzy $h_{\mathcal{I}}$-irresolute and fuzzy $h_{\mathcal{I}}$-open functions. Section 9 presents a concrete application to multi-criteria decision making. Section 10 concludes the paper.

2. Preliminaries

In this section, we present basic definitions and results that will be used throughout the paper. Let X be a non-empty set and I^X denote the collection of all fuzzy sets on X, i.e., all functions $\lambda\colon X \to [0,1]$.

For fuzzy sets $\lambda, \mu \in I^X$, we write $\lambda \leq \mu$ if $\lambda(x) \leq \mu(x)$ for all $x \in X$. The fuzzy sets $\mathbf{0}$ and $\mathbf{1}$ denote the constant functions 0 and 1 on X, respectively. For $\lambda, \mu \in I^X$, we define:

- $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$ (fuzzy union);
- $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$ (fuzzy intersection);
- $\lambda'(x) = 1 - \lambda(x)$ (fuzzy complement).

More generally, for a family $\{\lambda_\alpha\}_{\alpha\in\Delta} \subseteq I^X$:

$$\Big(\bigvee_{\alpha\in\Delta} \lambda_\alpha\Big)(x) = \sup_{\alpha\in\Delta} \lambda_\alpha(x) \quad \text{and} \quad \Big(\bigwedge_{\alpha\in\Delta} \lambda_\alpha\Big)(x) = \inf_{\alpha\in\Delta} \lambda_\alpha(x).$$

Definition 1 ([4]). *A fuzzy topology on X is a family $\tilde{\tau} \subseteq I^X$ satisfying:*

(i) $\mathbf{0}, \mathbf{1} \in \tilde{\tau}$;
(ii) *If $\lambda_1, \lambda_2 \in \tilde{\tau}$, then $\lambda_1 \wedge \lambda_2 \in \tilde{\tau}$;*
(iii) *If $\{\lambda_\alpha\}_{\alpha\in\Delta} \subseteq \tilde{\tau}$, then $\bigvee_{\alpha\in\Delta} \lambda_\alpha \in \tilde{\tau}$.*

The pair $(X, \tilde{\tau})$ is called a fuzzy topological space. Members of $\tilde{\tau}$ are called fuzzy open sets, and their complements are called fuzzy closed sets.

For a fuzzy set λ in $(X, \tilde{\tau})$, the fuzzy interior and fuzzy closure of λ are defined by

$$\text{int}(\lambda) = \bigvee\{\mu \in \tilde{\tau} : \mu \leq \lambda\}, \qquad \text{cl}(\lambda) = \bigwedge\{\mu' : \mu \in \tilde{\tau},\, \lambda \leq \mu'\},$$

where $\mu' = \mathbf{1} - \mu$ is the complement. Equivalently, $\text{cl}(\lambda) = \mathbf{1} - \text{int}(\mathbf{1} - \lambda)$.

Definition 2 ([17]). *A non-empty subcollection $\tilde{\mathcal{I}} \subseteq I^X$ is called a fuzzy ideal on X if:*

(i) *(Heredity) If $\lambda \in \tilde{\mathcal{I}}$ and $\mu \leq \lambda$, then $\mu \in \tilde{\mathcal{I}}$;*
(ii) *(Finite additivity) If $\lambda, \mu \in \tilde{\mathcal{I}}$, then $\lambda \vee \mu \in \tilde{\mathcal{I}}$.*

A fuzzy ideal topological space is a triple $(X, \tilde{\tau}, \tilde{\mathcal{I}})$, where $\tilde{\tau}$ is a fuzzy topology and $\tilde{\mathcal{I}}$ is a fuzzy ideal on X.

Definition 3 ([17]). *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda \in I^X$. The fuzzy local function of λ with respect to $\tilde{\mathcal{I}}$ and $\tilde{\tau}$, denoted by $\lambda^*(\tilde{\mathcal{I}}, \tilde{\tau})$ (or simply λ^*), is the fuzzy set defined by*

$$\lambda^*(\tilde{\mathcal{I}}, \tilde{\tau})(x) = \bigwedge\{(\mathbf{1} - \mu)(x) : \mu \in \tilde{\tau},\ \lambda \wedge \mu \in \tilde{\mathcal{I}}\}$$

for each $x \in X$.

Remark 1. *Since $\mathbf{0} \in \tilde{\tau}$ and $\lambda \wedge \mathbf{0} = \mathbf{0} \in \tilde{\mathcal{I}}$, the set $\{\mu \in \tilde{\tau} : \lambda \wedge \mu \in \tilde{\mathcal{I}}\}$ is always non-empty, ensuring that λ^* is well-defined. Moreover, $\lambda^*(x) \le (\mathbf{1} - \mathbf{0})(x) = 1$ and $\lambda^* \ge \mathbf{0}$.*

The *fuzzy $*$-closure operator* is defined by $\mathrm{Cl}^*(\lambda) = \lambda \vee \lambda^*$ for each $\lambda \in I^X$. It is known [17] that Cl^* is a Kuratowski fuzzy closure operator that generates a fuzzy topology $\tilde{\tau}^* = \tilde{\tau}^*(\tilde{\mathcal{I}})$ finer than $\tilde{\tau}$, defined by $\tilde{\tau}^* = \{\lambda \in I^X : \mathrm{Cl}^*(\mathbf{1} - \lambda) = \mathbf{1} - \lambda\}$.

The *fuzzy $*$-interior* is defined by $\mathrm{Int}^*(\lambda) = \mathbf{1} - \mathrm{Cl}^*(\mathbf{1} - \lambda)$ for each $\lambda \in I^X$.

We recall the following definitions of generalized fuzzy open sets:

Definition 4. *A fuzzy set λ in a fuzzy topological space $(X, \tilde{\tau})$ is said to be:*

(i) *Fuzzy α-open [6] if $\lambda \le \mathrm{int}(\mathrm{cl}(\mathrm{int}(\lambda)))$;*
(ii) *Fuzzy pre-open [6] if $\lambda \le \mathrm{int}(\mathrm{cl}(\lambda))$;*
(iii) *Fuzzy semi-open [5] if $\lambda \le \mathrm{cl}(\mathrm{int}(\lambda))$;*
(iv) *Fuzzy β-open [22] if $\lambda \le \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\lambda)))$.*

The *fuzzy α-interior* $\alpha\mathrm{int}(\lambda)$, *fuzzy pre-interior* $p\mathrm{int}(\lambda)$, *fuzzy semi-interior* $s\mathrm{int}(\lambda)$, and *fuzzy β-interior* $\beta\mathrm{int}(\lambda)$ of a fuzzy set λ are defined as the supremum of all fuzzy α-open (resp. pre-open, semi-open, β-open) sets contained in λ.

Definition 5 ([20]). *A fuzzy set λ in a fuzzy topological space $(X, \tilde{\tau})$ is said to be fuzzy h-open if*

$$\lambda \le \mathrm{int}(\lambda \vee \mu)$$

for all $\mu \in \tilde{\tau}$ such that $\mathbf{0} \ne \mu \ne \mathbf{1}$. The family of all fuzzy h-open sets in $(X, \tilde{\tau})$ is denoted by $\tilde{\tau}^h$.

3. Fuzzy $h_{\mathcal{I}}$-Open Sets

In this section, we introduce the main concept of this paper and establish its fundamental properties.

Definition 6. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space. A fuzzy set $\lambda \in I^X$ is said to be fuzzy $h_{\mathcal{I}}$-open if*

$$\lambda \le \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$$

for all $\mu \in \tilde{\tau}$ such that $\mathbf{0} \ne \mu \ne \mathbf{1}$. The complement of a fuzzy $h_{\mathcal{I}}$-open set is said to be fuzzy $h_{\mathcal{I}}$-closed. The family of all fuzzy $h_{\mathcal{I}}$-open sets of $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ is denoted by $\tilde{\tau}^{h_{\mathcal{I}}}$.

Remark 2. *In Definition 6, $\mathrm{Cl}^*(\mu) = \mu \vee \mu^*$ denotes the fuzzy $*$-closure of μ. The use of $\mathrm{Cl}^*(\mu)$ rather than simply μ incorporates the ideal-based correction into the definition, allowing the topology $\tilde{\tau}^{h_{\mathcal{I}}}$ to reflect the influence of the fuzzy ideal $\tilde{\mathcal{I}}$.*

Theorem 1. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda \in I^X$. If λ is fuzzy h-open, then λ is fuzzy $h_{\mathcal{I}}$-open.*

Proof. Let λ be a fuzzy h-open set. For any $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, we have $\lambda \leq \mathrm{int}(\lambda \vee \mu)$. Since $\mathrm{Cl}^*(\mu) = \mu \vee \mu^* \geq \mu$, it follows that $\lambda \vee \mathrm{Cl}^*(\mu) \geq \lambda \vee \mu$, and, by the monotonicity of int:

$$\lambda \leq \mathrm{int}(\lambda \vee \mu) \leq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu)).$$

Therefore, λ is fuzzy $h_{\mathcal{I}}$-open. □

The converse of Theorem 1 does not hold in general, as demonstrated by the following example:

Example 1. *Let $X = \{a, b, c\}$ and define the following fuzzy sets using characteristic functions: $\mu_1 = \chi_{\{b\}}$ and $\mu_2 = \chi_{\{b,c\}}$. Let $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2\}$ and $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a) = 0, \lambda(b) = 0\}$ (i.e., $\tilde{\mathcal{I}}$ consists of all fuzzy sets supported only on $\{c\}$).*

One can verify that the family of all fuzzy h-open sets is $\tilde{\tau}^h = \{\mathbf{0}, \mathbf{1}, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{a,c\}}, \chi_{\{b,c\}}\}$.

Now consider $\lambda = \chi_{\{a,b\}}$. For the non-trivial fuzzy open sets μ_1 and μ_2, we compute the following:

For $\mu_1^*(x)$, we need $\bigwedge\{1 - \nu(x) : \nu \in \tilde{\tau}, \mu_1 \wedge \nu \in \tilde{\mathcal{I}}\}$. Since $\mu_1(b) = 1 > 0$, for $\mu_1 \wedge \nu \in \tilde{\mathcal{I}}$, we need $(\mu_1 \wedge \nu)(b) = 0$; hence, $\nu(b) = 0$. Among elements of $\tilde{\tau}$, only $\mathbf{0}$ satisfies $\nu(b) = 0$. Therefore, $\mu_1^*(x) = 1 - \mathbf{0}(x) = 1$ for all x, i.e., $\mu_1^* = \mathbf{1}$ and $\mathrm{Cl}^*(\mu_1) = \mathbf{1}$.

Similarly, $\mathrm{Cl}^*(\mu_2) = \mathbf{1}$.

Therefore, $\mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu_i)) = \mathrm{int}(\lambda \vee \mathbf{1}) = \mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda$ for $i = 1, 2$. Hence, $\lambda = \chi_{\{a,b\}} \in \tilde{\tau}^{h_{\mathcal{I}}}$.

However, for the fuzzy h-open condition, $\mathrm{int}(\lambda \vee \mu_1) = \mathrm{int}(\chi_{\{a,b\}}) = \mu_1 = \chi_{\{b\}}$, and $\lambda(a) = 1 > 0 = \mu_1(a)$, so $\lambda \not\leq \mathrm{int}(\lambda \vee \mu_1)$. Thus, $\lambda \notin \tilde{\tau}^h$.

This demonstrates that $\tilde{\tau}^h \subsetneq \tilde{\tau}^{h_{\mathcal{I}}}$ in general.

Proposition 1. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space. If $\tilde{\mathcal{I}} = I^X$ (the maximal fuzzy ideal), then $\tilde{\tau}^h = \tilde{\tau}^{h_{\mathcal{I}}}$.*

Proof. If $\tilde{\mathcal{I}} = I^X$, then, for every fuzzy set λ, $\lambda \wedge \mathbf{1} = \lambda \in \tilde{\mathcal{I}}$, which gives $\lambda^*(x) \leq 1 - \mathbf{1}(x) = 0$ for all x. Hence, $\lambda^* = \mathbf{0}$ for all $\lambda \in I^X$. In particular, for any $\mu \in \tilde{\tau}$, $\mathrm{Cl}^*(\mu) = \mu \vee \mu^* = \mu \vee \mathbf{0} = \mu$.

Now let λ be any fuzzy $h_{\mathcal{I}}$-open set. Then, $\lambda \leq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu)) = \mathrm{int}(\lambda \vee \mu)$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, which means λ is fuzzy h-open. Together with Theorem 1, we obtain $\tilde{\tau}^h = \tilde{\tau}^{h_{\mathcal{I}}}$. □

The following theorem is one of the main results of this paper:

Theorem 2. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space. Then, the family $\tilde{\tau}^{h_{\mathcal{I}}}$ of all fuzzy $h_{\mathcal{I}}$-open sets forms a fuzzy topology on X.*

Proof. (1) It is clear that $\mathbf{0}, \mathbf{1} \in \tilde{\tau}^{h_{\mathcal{I}}}$. Indeed, $\mathbf{0} \leq \mathrm{int}(\mathbf{0} \vee \mathrm{Cl}^*(\mu)) = \mathrm{int}(\mathrm{Cl}^*(\mu))$ trivially (since $\mathbf{0}$ is the minimum), and $\mathbf{1} \leq \mathrm{int}(\mathbf{1} \vee \mathrm{Cl}^*(\mu)) = \mathrm{int}(\mathbf{1}) = \mathbf{1}$.

(2) Let $\lambda_1, \lambda_2 \in \tilde{\tau}^{h_{\mathcal{I}}}$. We show that $\lambda_1 \wedge \lambda_2 \in \tilde{\tau}^{h_{\mathcal{I}}}$. Let $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$. Then, $\lambda_1 \leq \mathrm{int}(\lambda_1 \vee \mathrm{Cl}^*(\mu))$ and $\lambda_2 \leq \mathrm{int}(\lambda_2 \vee \mathrm{Cl}^*(\mu))$. Therefore,

$$\begin{aligned}\lambda_1 \wedge \lambda_2 &\leq \mathrm{int}(\lambda_1 \vee \mathrm{Cl}^*(\mu)) \wedge \mathrm{int}(\lambda_2 \vee \mathrm{Cl}^*(\mu)) \\ &= \mathrm{int}\big((\lambda_1 \vee \mathrm{Cl}^*(\mu)) \wedge (\lambda_2 \vee \mathrm{Cl}^*(\mu))\big) \\ &= \mathrm{int}\big((\lambda_1 \wedge \lambda_2) \vee \mathrm{Cl}^*(\mu)\big),\end{aligned}$$

where the second equality uses the fact that $\mathrm{int}(\alpha) \wedge \mathrm{int}(\beta) = \mathrm{int}(\alpha \wedge \beta)$ for any fuzzy sets α, β, and the third equality uses the distributive law in I^X. Hence, $\lambda_1 \wedge \lambda_2 \in \tilde{\tau}^{h_{\mathcal{I}}}$.

(3) Let $\{\lambda_\alpha\}_{\alpha\in\Delta} \subseteq \tilde{\tau}^{h_\mathcal{I}}$. We show that $\bigvee_{\alpha\in\Delta}\lambda_\alpha \in \tilde{\tau}^{h_\mathcal{I}}$. For each $\alpha \in \Delta$ and all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, we have $\lambda_\alpha \leq \mathrm{int}(\lambda_\alpha \vee \mathrm{Cl}^*(\mu))$. Since $\lambda_\alpha \leq \bigvee_{\alpha\in\Delta}\lambda_\alpha$, it follows that $\lambda_\alpha \vee \mathrm{Cl}^*(\mu) \leq (\bigvee_\alpha \lambda_\alpha) \vee \mathrm{Cl}^*(\mu)$, and hence

$$\lambda_\alpha \leq \mathrm{int}(\lambda_\alpha \vee \mathrm{Cl}^*(\mu)) \leq \mathrm{int}\big((\textstyle\bigvee_\alpha \lambda_\alpha) \vee \mathrm{Cl}^*(\mu)\big).$$

Taking the supremum over all $\alpha \in \Delta$:

$$\bigvee_{\alpha\in\Delta} \lambda_\alpha \leq \mathrm{int}\big((\textstyle\bigvee_\alpha \lambda_\alpha) \vee \mathrm{Cl}^*(\mu)\big).$$

Therefore, $\bigvee_{\alpha\in\Delta}\lambda_\alpha \in \tilde{\tau}^{h_\mathcal{I}}$. □

The next result establishes the precise relationship between the fuzzy topologies $\tilde{\tau}$, $\tilde{\tau}^*$, and $\tilde{\tau}^{h_\mathcal{I}}$. Unlike extensions of open sets that typically refine $\tilde{\tau}^*$, the $h_\mathcal{I}$-topology is generally *incomparable* with $\tilde{\tau}^*$, showing that the two constructions capture fundamentally different aspects of the interaction between the topology and the ideal.

Theorem 3. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space. Then, $\tilde{\tau} \subseteq \tilde{\tau}^{h_\mathcal{I}}$ and $\tilde{\tau} \subseteq \tilde{\tau}^*$, but $\tilde{\tau}^*$ and $\tilde{\tau}^{h_\mathcal{I}}$ are in general incomparable; neither $\tilde{\tau}^* \subseteq \tilde{\tau}^{h_\mathcal{I}}$ nor $\tilde{\tau}^{h_\mathcal{I}} \subseteq \tilde{\tau}^*$ holds in general.*

Proof. The inclusion $\tilde{\tau} \subseteq \tilde{\tau}^{h_\mathcal{I}}$ follows from Theorem 1, since every fuzzy open set is fuzzy h-open and every fuzzy h-open set is fuzzy $h_\mathcal{I}$-open. The inclusion $\tilde{\tau} \subseteq \tilde{\tau}^*$ is well-known [17]. The incomparability is demonstrated by Examples 2 and 3 below. □

Example 2. ***($\tilde{\tau}^* \nsubseteq \tilde{\tau}^{h_\mathcal{I}}$).*** *Let $X = \{a, b, c\}$, $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \chi_{\{a\}}, \chi_{\{b,c\}}\}$, and $\tilde{\mathcal{I}} = \{\lambda \in I^X : \mathrm{supp}(\lambda) \subseteq \{c\}\}$. One can verify that $\mathrm{Cl}^*(\chi_{\{a\}}) = \chi_{\{a\}}$ and $\mathrm{Cl}^*(\chi_{\{b,c\}}) = \chi_{\{b,c\}}$.*

Consider $\lambda = \chi_{\{a,b\}}$. To check $\lambda \in \tilde{\tau}^*$, we compute $\mathrm{Cl}^*(\mathbf{1} - \lambda) = \mathrm{Cl}^*(\chi_{\{c\}})$. Since $\mathrm{supp}(\chi_{\{c\}}) \subseteq \{c\}$ and all qualifying $\nu \in \tilde{\tau}$ (with $\chi_{\{c\}} \wedge \nu \in \tilde{\mathcal{I}}$) include $\mathbf{0}$, $\chi_{\{a\}}$, $\chi_{\{b,c\}}$, and $\mathbf{1}$ (since $\chi_{\{c\}} \wedge \nu$ always has support in $\{c\}$), we obtain $\chi^*_{\{c\}} = \mathbf{0}$ and $\mathrm{Cl}^*(\chi_{\{c\}}) = \chi_{\{c\}}$. Hence, $\lambda \in \tilde{\tau}^*$.

However, λ is not fuzzy $h_\mathcal{I}$-open. For $\mu = \chi_{\{a\}}$, $\mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu)) = \mathrm{int}(\chi_{\{a,b\}} \vee \chi_{\{a\}}) = \mathrm{int}(\chi_{\{a,b\}}) = \chi_{\{a\}}$, and $\lambda(b) = 1 > 0 = \chi_{\{a\}}(b)$, so $\lambda \nleq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$. Therefore, $\chi_{\{a,b\}} \in \tilde{\tau}^* \setminus \tilde{\tau}^{h_\mathcal{I}}$.

Example 3. ***($\tilde{\tau}^{h_\mathcal{I}} \nsubseteq \tilde{\tau}^*$).*** *Using the space in Example 1, $X = \{a, b, c\}$, $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \chi_{\{b\}}, \chi_{\{b,c\}}\}$, $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a) = 0,\ \lambda(b) = 0\}$. We showed that $\mathrm{Cl}^*(\mu) = \mathbf{1}$ for the non-trivial opens, so $\tilde{\tau}^{h_\mathcal{I}} = I^X$ (the discrete fuzzy topology).*

However, consider $\lambda(a) = 0.5$, $\lambda(b) = 0.5$, $\lambda(c) = 0.5$. Then, $\mathbf{1} - \lambda = (0.5, 0.5, 0.5)$. Computing $(\mathbf{1} - \lambda)^*$, the qualifying $\nu \in \tilde{\tau}$ (with $(\mathbf{1} - \lambda) \wedge \nu \in \tilde{\mathcal{I}}$) must satisfy $\min\{0.5, \nu(a)\} = 0$ and $\min\{0.5, \nu(b)\} = 0$, i.e., $\nu(a) = 0$ and $\nu(b) = 0$. Only $\nu = \mathbf{0}$ qualifies. Hence, $(\mathbf{1} - \lambda)^* = \mathbf{1}$ and $\mathrm{Cl}^*(\mathbf{1} - \lambda) = \mathbf{1} \neq \mathbf{1} - \lambda$. Therefore, $\lambda \notin \tilde{\tau}^*$, while $\lambda \in \tilde{\tau}^{h_\mathcal{I}} = I^X$.

Remark 3. *If $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}\}$ (the indiscrete fuzzy topology), then the condition in Definition 6 is vacuously satisfied (since there is no $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$), so $\tilde{\tau}^{h_\mathcal{I}} = I^X$ regardless of the ideal. This shows that the $h_\mathcal{I}$-construction is most informative when $\tilde{\tau}$ has a sufficiently rich family of open sets.*

The following example demonstrates the theory with genuinely fuzzy (non-characteristic) membership values, illustrating that the $h_\mathcal{I}$-construction captures nuanced interactions between partial membership degrees and ideal membership that have no crisp counterpart.

Example 4. *Let $X = \{a, b\}$ and define fuzzy sets $\mu_1(a) = 0.3$, $\mu_1(b) = 0.7$ and $\mu_2(a) = 0.6$, and $\mu_2(b) = 0.4$. Set $\mu_3 = \mu_1 \wedge \mu_2 = (0.3, 0.4)$ and $\mu_4 = \mu_1 \vee \mu_2 = (0.6, 0.7)$. Let $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2, \mu_3, \mu_4\}$ and $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a) \leq 0.2, \lambda(b) \leq 0.3\}$.*

Step 1: Computing Cl^* for non-trivial opens. *For $\mu_1 = (0.3, 0.7)$, the qualifying $\nu \in \tilde{\tau}$ (with $\mu_1 \wedge \nu \in \tilde{\mathcal{I}}$) must satisfy $\min\{0.3, \nu(a)\} \leq 0.2$ and $\min\{0.7, \nu(b)\} \leq 0.3$. From $\tilde{\tau}$, only $\nu = \mathbf{0}$ (giving $(0, 0) \in \tilde{\mathcal{I}}$) qualifies, since $\mu_1 \wedge \mu_3 = \mu_3 = (0.3, 0.4)$ has $0.3 > 0.2$. Hence, $\mu_1^* = \mathbf{1}$ and $\mathrm{Cl}^*(\mu_1) = \mathbf{1}$.*

Similarly, one verifies $\mathrm{Cl}^*(\mu_2) = \mathbf{1}$, $\mathrm{Cl}^*(\mu_4) = \mathbf{1}$, and $\mathrm{Cl}^*(\mu_3) = \mathbf{1}$ (since $\mu_3 \wedge \mu_3 = \mu_3 = (0.3, 0.4) \notin \tilde{\mathcal{I}}$, only $\nu = \mathbf{0}$ qualifies for each).

Step 2: Determining $\tilde{\tau}^{h_{\mathcal{I}}}$. *Since $\mathrm{Cl}^*(\mu) = \mathbf{1}$ for every non-trivial $\mu \in \tilde{\tau}$, for any $\lambda \in I^X$, $\mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu)) = \mathrm{int}(\lambda \vee \mathbf{1}) = \mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda$. Hence, $\tilde{\tau}^{h_{\mathcal{I}}} = I^X$.*

Step 3: Verifying $\tilde{\tau}^* \subsetneq \tilde{\tau}^{h_{\mathcal{I}}}$. *Consider $\lambda = (0.5, 0.5)$. Then $(\mathbf{1} - \lambda)^*$: for $(0.5, 0.5) \wedge \nu \in \tilde{\mathcal{I}}$, we need $\min\{0.5, \nu(a)\} \leq 0.2$ and $\min\{0.5, \nu(b)\} \leq 0.3$, which requires $\nu(a) \leq 0.2$ and $\nu(b) \leq 0.3$. Only $\nu = \mathbf{0}$ qualifies. Hence, $(\mathbf{1} - \lambda)^* = \mathbf{1}$, $\mathrm{Cl}^*(\mathbf{1} - \lambda) = \mathbf{1} \neq \mathbf{1} - \lambda$, and $\lambda \notin \tilde{\tau}^*$. Since $\lambda \in \tilde{\tau}^{h_{\mathcal{I}}} = I^X$, this confirms $\tilde{\tau}^* \subsetneq \tilde{\tau}^{h_{\mathcal{I}}}$ with genuinely fuzzy witness.*

This example illustrates a key phenomenon: when the ideal is "small" relative to the topology (i.e., the $*$-closure operator inflates sets dramatically), the $h_{\mathcal{I}}$-topology can become much larger than $\tilde{\tau}^*$, detecting openness conditions that the $*$-topology misses.

4. Generalizations of Fuzzy $h_{\mathcal{I}}$-Open Sets

By replacing the standard fuzzy interior with various generalized interiors, we obtain a hierarchy of generalized fuzzy $h_{\mathcal{I}}$-open sets.

Definition 7. *A fuzzy set λ in a fuzzy ideal topological space $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ is said to be:*

(i) Fuzzy $h\alpha_{\mathcal{I}}$-open if $\lambda \leq \alpha\mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, where $\alpha\mathrm{int}$ is the fuzzy α-interior (Definition 4);

(ii) Fuzzy $hp_{\mathcal{I}}$-open if $\lambda \leq \mathrm{pint}(\lambda \vee \mathrm{Cl}^*(\mu))$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, where pint is the fuzzy pre-interior (Definition 4);

(iii) Fuzzy $hs_{\mathcal{I}}$-open if $\lambda \leq \mathrm{sint}(\lambda \vee \mathrm{Cl}^*(\mu))$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, where sint is the fuzzy semi-interior (Definition 4);

(iv) Fuzzy $h\beta_{\mathcal{I}}$-open if $\lambda \leq \beta\mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$, where $\beta\mathrm{int}$ is the fuzzy β-interior (Definition 4).

Remark 4. *The symbols α, p, s, and β in Definition 7 are not arbitrary parameters or variable names; they refer to the four specific generalized interior operators recalled in Definition 4:*

$$\alpha\mathrm{int}(\lambda) = \bigvee\{\mu \in I^X : \mu \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mu))),\ \mu \leq \lambda\} \qquad \textit{(fuzzy } \alpha\textit{-interior)},$$
$$\mathrm{pint}(\lambda) = \bigvee\{\mu \in I^X : \mu \leq \mathrm{int}(\mathrm{cl}(\mu)),\ \mu \leq \lambda\} \qquad \textit{(fuzzy pre-interior)},$$
$$\mathrm{sint}(\lambda) = \bigvee\{\mu \in I^X : \mu \leq \mathrm{cl}(\mathrm{int}(\mu)),\ \mu \leq \lambda\} \qquad \textit{(fuzzy semi-interior)},$$
$$\beta\mathrm{int}(\lambda) = \bigvee\{\mu \in I^X : \mu \leq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mu))),\ \mu \leq \lambda\} \qquad \textit{(fuzzy } \beta\textit{-interior)}.$$

These four operators arise from the classical Levine–Mashhour–Njåstad hierarchy by composing int and cl in all possible alternating sequences of length at most three. Since $\mathrm{int} \circ \mathrm{int} = \mathrm{int}$ and $\mathrm{cl} \circ \mathrm{cl} = \mathrm{cl}$, longer alternating compositions reduce to one of these four cases. Consequently, no other generalized interiors of this type exist within this scheme, which is precisely why exactly these four classes appear in Definition 7.

Example 5. *Let $X = \{a, b, c\}$ and define fuzzy sets $\mu_1 = (0.4, 0.7, 0.2)$, $\mu_2 = (0.6, 0.3, 0.8)$. Set $\mu_3 = \mu_1 \wedge \mu_2 = (0.4, 0.3, 0.2)$ and $\mu_4 = \mu_1 \vee \mu_2 = (0.6, 0.7, 0.8)$. Let $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2, \mu_3, \mu_4\}$ and $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(x) \leq 0.1$ for all $x\}$.*

Step 1: Computing $\mathbf{Cl}^*$**.** *Since* $\tilde{\mathcal{I}}$ *is very small, for each non-trivial* $\mu \in \tilde{\tau}$*, the condition* $\mu \wedge \nu \in \tilde{\mathcal{I}}$ *(requiring all values* ≤ 0.1*) forces* $\nu = \mathbf{0}$*. Hence,* $\mu^* = \mathbf{1}$ *and* $\mathrm{Cl}^*(\mu) = \mathbf{1}$ *for every non-trivial open set.*

Step 2: Checking the four types. *Consider* $\lambda = (0.5, 0.4, 0.6)$*. For each non-trivial* $\mu \in \tilde{\tau}$*,* $\lambda \vee \mathrm{Cl}^*(\mu) = \lambda \vee \mathbf{1} = \mathbf{1}$*. Therefore:*

$$\mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda, \quad \alpha\mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda, \quad p\mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda,$$
$$s\mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda, \quad \beta\mathrm{int}(\mathbf{1}) = \mathbf{1} \geq \lambda.$$

Hence, λ *is simultaneously fuzzy* $h_{\mathcal{I}}$*-open,* $h\alpha_{\mathcal{I}}$*-open,* $hp_{\mathcal{I}}$*-open,* $hs_{\mathcal{I}}$*-open, and* $h\beta_{\mathcal{I}}$*-open. This illustrates how to check membership in each class using the corresponding generalized interior from Definition 4.*

Step 3: When the classes differ. *With a small ideal, the four classes may collapse. However, when the ideal is taken to be maximal (*$\tilde{\mathcal{I}} = I^X$*), Proposition 3.5 gives* $\tilde{\tau}^{h_{\mathcal{I}}} = \tilde{\tau}^h$*, and the* $h\alpha_{\mathcal{I}}$*,* $hp_{\mathcal{I}}$*,* $hs_{\mathcal{I}}$*,* $h\beta_{\mathcal{I}}$ *classes reduce to their non-ideal counterparts, which are known to be pairwise distinct from the classical theory. Examples 6 and 7 below demonstrate this distinctness explicitly.*

From Definition 7 and the well-known inclusions among generalized fuzzy interiors, we have the following diagram of implications:

$$\begin{array}{ccccccc} \text{fuzzy } h\text{-open} & \Rightarrow & \text{fuzzy } h_{\mathcal{I}}\text{-open} & \Rightarrow & \text{fuzzy } h\alpha_{\mathcal{I}}\text{-open} & \Rightarrow & \text{fuzzy } hp_{\mathcal{I}}\text{-open} \\ & & & & \Downarrow & & \Downarrow \\ & & & & \text{fuzzy } hs_{\mathcal{I}}\text{-open} & \Rightarrow & \text{fuzzy } h\beta_{\mathcal{I}}\text{-open} \end{array}$$

Remark 5. *The converses of the above implications are not true in general.*

Example 6. *Let* $X = \{a, b, c, d\}$*. Define fuzzy open sets by their characteristic functions:* $\mu_1 = \chi_{\{a\}}$*,* $\mu_2 = \chi_{\{c\}}$*,* $\mu_3 = \chi_{\{a,c\}}$*,* $\mu_4 = \chi_{\{a,d\}}$*, and* $\mu_5 = \chi_{\{a,c,d\}}$*. Let* $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ *and* $\tilde{\mathcal{I}} = \{\lambda \in I^X : \mathrm{supp}(\lambda) \subseteq \{a, c\}\}$*. Then, one can verify that* $\chi_{\{a,b,c\}}$ *is fuzzy* $h\alpha_{\mathcal{I}}$*-open but not fuzzy* $h_{\mathcal{I}}$*-open.*

Example 7. *Let* $X = \{a, b, c, d\}$*. Define* $\mu_1 = \chi_{\{d\}}$*,* $\mu_2 = \chi_{\{a,c\}}$*,* $\mu_3 = \chi_{\{a,c,d\}}$*. Let* $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2, \mu_3\}$ *and* $\tilde{\mathcal{I}} = \{\lambda \in I^X : \mathrm{supp}(\lambda) \subseteq \{a\}\}$*. Then,* $\chi_{\{b,d\}}$ *is fuzzy* $hs_{\mathcal{I}}$*-open but not fuzzy* $hp_{\mathcal{I}}$*-open, and* $\chi_{\{c,d\}}$ *is fuzzy* $hp_{\mathcal{I}}$*-open but not fuzzy* $hs_{\mathcal{I}}$*-open. This demonstrates that the classes of fuzzy* $hs_{\mathcal{I}}$*-open sets and fuzzy* $hp_{\mathcal{I}}$*-open sets are independent.*

The following genuinely fuzzy example demonstrates that the hierarchy captures nuanced distinctions visible only at non-trivial membership degrees.

Example 8. *Let* $X = \{a, b, c\}$ *and define fuzzy sets* $\mu_1 = (0.4, 0.6, 0.2)$*,* $\mu_2 = (0.7, 0.3, 0.5)$*,* $\mu_3 = \mu_1 \wedge \mu_2 = (0.4, 0.3, 0.2)$*,* $\mu_4 = \mu_1 \vee \mu_2 = (0.7, 0.6, 0.5)$*. Let* $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \mu_1, \mu_2, \mu_3, \mu_4\}$ *and* $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a) \leq 0.1,\ \lambda(b) \leq 0.1,\ \lambda(c) \leq 0.1\}$*.*

Since $\tilde{\mathcal{I}}$ *is a very small ideal (only nearly-zero fuzzy sets are negligible),* $\mu_i \wedge \nu \in \tilde{\mathcal{I}}$ *requires* ν *to be extremely small on* $\mathrm{supp}(\mu_i)$*. For each non-trivial* μ_i*, only* $\nu = \mathbf{0}$ *satisfies the condition, yielding* $\mu_i^* = \mathbf{1}$ *and* $\mathrm{Cl}^*(\mu_i) = \mathbf{1}$*.*

Consider $\lambda = (0.5, 0.5, 0.3)$*. Since* $\mathrm{Cl}^*(\mu) = \mathbf{1}$ *for all non-trivial* μ*:*

(i) $\mathrm{int}(\lambda \vee \mathbf{1}) = \mathbf{1} \geq \lambda$*, so* λ *is fuzzy* $h_{\mathcal{I}}$*-open (hence also* $h\alpha_{\mathcal{I}}$*-open,* $hp_{\mathcal{I}}$*-open,* $hs_{\mathcal{I}}$*-open, and* $h\beta_{\mathcal{I}}$*-open).*

Now consider a different ideal $\tilde{\mathcal{I}}' = \{\lambda \in I^X : \lambda(a) \leq 0.5,\ \lambda(b) \leq 0.5,\ \lambda(c) \leq 0.5\}$*. With this larger ideal, for* $\mu_3 = (0.4, 0.3, 0.2)$*,* $\mu_3 \in \tilde{\mathcal{I}}'$*, so* $\mu_3 \wedge \mu_3 = \mu_3 \in \tilde{\mathcal{I}}'$*, meaning* $\nu = \mu_3$ *qualifies and* $\mu_3^*(x) \leq 1 - \mu_3(x)$*. One computes* $\mathrm{Cl}^*(\mu_3) = (0.6, 0.7, 0.8)$ *(not* $\mathbf{1}$*).*

https://doi.org/10.3390/math14050904

For $\lambda = (0.5, 0.5, 0.3)$, $\text{int}(\lambda \vee \text{Cl}^*(\mu_3)) = \text{int}((0.6, 0.7, 0.8)) = \mu_4 = (0.7, 0.6, 0.5) \geq \lambda$, so λ is still $h_{\mathcal{I}'}$-open. However, for $\gamma = (0.8, 0.1, 0.6)$, $\text{int}(\gamma \vee \text{Cl}^*(\mu_3)) = \text{int}((0.8, 0.7, 0.8)) = \mu_4 = (0.7, 0.6, 0.5)$ and $\gamma(a) = 0.8 > 0.7$, so γ is not $h_{\mathcal{I}'}$-open. However, $\alpha\text{int}(\gamma \vee \text{Cl}^*(\mu_3)) = \alpha\text{int}((0.8, 0.7, 0.8))$ may still satisfy $\gamma \leq \alpha\text{int}(\gamma \vee \text{Cl}^*(\mu_3))$ depending on the α-interior computation, making γ potentially $h\alpha_{\mathcal{I}'}$-open but not $h_{\mathcal{I}'}$-open.

This example illustrates how the size of the ideal controls the fineness of the hierarchy: a small ideal tends to collapse the classes (all become discrete), while a larger ideal preserves meaningful distinctions between the generalized openness levels.

We summarize the relationships in Table 1.

Table 1. Summary of generalized fuzzy openness classes, their notation, and the interior operators used in their definitions.

Class	Notation	Interior Used
Fuzzy h-open	$\tilde{\tau}^h$	int with μ (no ideal)
Fuzzy $h_{\mathcal{I}}$-open	$\tilde{\tau}^{h_{\mathcal{I}}}$	int with $\text{Cl}^*(\mu)$
Fuzzy $h\alpha_{\mathcal{I}}$-open	—	αint with $\text{Cl}^*(\mu)$
Fuzzy $hp_{\mathcal{I}}$-open	—	$p$int with $\text{Cl}^*(\mu)$
Fuzzy $hs_{\mathcal{I}}$-open	—	$s$int with $\text{Cl}^*(\mu)$
Fuzzy $h\beta_{\mathcal{I}}$-open	—	βint with $\text{Cl}^*(\mu)$

The strict implications are: h-open $\Rightarrow$ $h_{\mathcal{I}}$-open $\Rightarrow$ $h\alpha_{\mathcal{I}}$-open $\Rightarrow$ $hp_{\mathcal{I}}$-open $\Rightarrow$ $h\beta_{\mathcal{I}}$-open and $h\alpha_{\mathcal{I}}$-open $\Rightarrow$ $hs_{\mathcal{I}}$-open $\Rightarrow$ $h\beta_{\mathcal{I}}$-open, with $hp_{\mathcal{I}}$-open and $hs_{\mathcal{I}}$-open independent.

5. Fuzzy $h_{\mathcal{I}}$-Interior and Fuzzy $h_{\mathcal{I}}$-Closure

Definition 8. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda \in I^X$. The fuzzy $h_{\mathcal{I}}$-interior of λ is defined by*

$$\text{Int}_{h_{\mathcal{I}}}(\lambda) = \bigvee \{\nu \in \tilde{\tau}^{h_{\mathcal{I}}} : \nu \leq \lambda\}.$$

Theorem 4. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda, \mu \in I^X$. Then:*

(i) *If $\lambda \leq \mu$, then $\text{Int}_{h_{\mathcal{I}}}(\lambda) \leq \text{Int}_{h_{\mathcal{I}}}(\mu)$.*
(ii) $\text{Int}_{h_{\mathcal{I}}}(\lambda) \leq \lambda$.
(iii) $\text{Int}_{h_{\mathcal{I}}}(\text{Int}_{h_{\mathcal{I}}}(\lambda)) = \text{Int}_{h_{\mathcal{I}}}(\lambda)$.
(iv) *λ is fuzzy $h_{\mathcal{I}}$-open if and only if $\lambda = \text{Int}_{h_{\mathcal{I}}}(\lambda)$.*
(v) $\text{Int}_{h_{\mathcal{I}}}(\lambda) \wedge \text{Int}_{h_{\mathcal{I}}}(\mu) = \text{Int}_{h_{\mathcal{I}}}(\lambda \wedge \mu)$.
(vi) $\text{Int}_{h_{\mathcal{I}}}(\lambda) \vee \text{Int}_{h_{\mathcal{I}}}(\mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda \vee \mu)$.

Proof. Since $\tilde{\tau}^{h_{\mathcal{I}}}$ is a fuzzy topology by Theorem 2, the operator $\text{Int}_{h_{\mathcal{I}}}$ is the standard interior operator of this topology. Parts (i)–(iv) follow from the general properties of interior operators in fuzzy topological spaces [4].

(v) ($\leq$) Since $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, by (i), $\text{Int}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda)$ and $\text{Int}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \text{Int}_{h_{\mathcal{I}}}(\mu)$. Hence, $\text{Int}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda) \wedge \text{Int}_{h_{\mathcal{I}}}(\mu)$. ($\geq$) Since $\text{Int}_{h_{\mathcal{I}}}(\lambda) \leq \lambda$ and $\text{Int}_{h_{\mathcal{I}}}(\mu) \leq \mu$, we have $\text{Int}_{h_{\mathcal{I}}}(\lambda) \wedge \text{Int}_{h_{\mathcal{I}}}(\mu) \leq \lambda \wedge \mu$. Moreover, $\text{Int}_{h_{\mathcal{I}}}(\lambda) \wedge \text{Int}_{h_{\mathcal{I}}}(\mu) \in \tilde{\tau}^{h_{\mathcal{I}}}$ (finite intersection). Therefore, $\text{Int}_{h_{\mathcal{I}}}(\lambda) \wedge \text{Int}_{h_{\mathcal{I}}}(\mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda \wedge \mu)$.

(vi) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, by (i), $\text{Int}_{h_{\mathcal{I}}}(\lambda) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda \vee \mu)$ and $\text{Int}_{h_{\mathcal{I}}}(\mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda \vee \mu)$, whence $\text{Int}_{h_{\mathcal{I}}}(\lambda) \vee \text{Int}_{h_{\mathcal{I}}}(\mu) \leq \text{Int}_{h_{\mathcal{I}}}(\lambda \vee \mu)$. Equality does not hold in general since $\tilde{\tau}^{h_{\mathcal{I}}}$, like any topology, is closed under finite intersection but not under finite union of interiors. □

Definition 9. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $x \in X$. A fuzzy $h_{\mathcal{I}}$-open set ν satisfying $\nu(x) > 0$ is called a fuzzy $h_{\mathcal{I}}$-open neighborhood of x.*

Definition 10. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $x \in X$. A fuzzy set $\nu \in I^X$ is called a fuzzy $h_{\mathcal{I}}$-neighborhood of x if there exists $\gamma \in \tilde{\tau}^{h_{\mathcal{I}}}$ such that $\gamma(x) > 0$ and $\gamma \leq \nu$.*

Remark 6. *Every fuzzy $h_{\mathcal{I}}$-open neighborhood is a fuzzy $h_{\mathcal{I}}$-neighborhood, but the converse is not always true.*

Definition 11. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda \in I^X$. The fuzzy $h_{\mathcal{I}}$-closure of λ is defined by*

$$\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) = \bigwedge\{\nu : \nu \text{ is fuzzy } h_{\mathcal{I}}\text{-closed}, \lambda \leq \nu\}.$$

Theorem 5. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda, \mu \in I^X$. Then:*

(i) If $\lambda \leq \mu$, then $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$.
(ii) $\lambda \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda)$.
(iii) $\mathrm{cl}_{h_{\mathcal{I}}}(\mathrm{cl}_{h_{\mathcal{I}}}(\lambda)) = \mathrm{cl}_{h_{\mathcal{I}}}(\lambda)$.
(iv) λ is fuzzy $h_{\mathcal{I}}$-closed if and only if $\lambda = \mathrm{cl}_{h_{\mathcal{I}}}(\lambda)$.
(v) $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \wedge \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$.
(vi) $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \vee \mathrm{cl}_{h_{\mathcal{I}}}(\mu) = \mathrm{cl}_{h_{\mathcal{I}}}(\lambda \vee \mu)$.

Proof. Since $\mathrm{cl}_{h_{\mathcal{I}}}$ is the closure operator dual to $\mathrm{Int}_{h_{\mathcal{I}}}$ in the fuzzy topology $\tilde{\tau}^{h_{\mathcal{I}}}$, parts (i)–(iv) follow from the general properties of closure operators in fuzzy topological spaces [4].

(v) Since $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, by (i), $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda)$ and $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$. Hence, $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda \wedge \mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \wedge \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$.

(vi) ($\leq$) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$, by (i), $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda \vee \mu)$ and $\mathrm{cl}_{h_{\mathcal{I}}}(\mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda \vee \mu)$; therefore, $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \vee \mathrm{cl}_{h_{\mathcal{I}}}(\mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda \vee \mu)$. ($\geq$) The set $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \vee \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$ is fuzzy $h_{\mathcal{I}}$-closed (finite union of closed sets in a topology) and $\lambda \vee \mu \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \vee \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$; therefore, $\mathrm{cl}_{h_{\mathcal{I}}}(\lambda \vee \mu) \leq \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) \vee \mathrm{cl}_{h_{\mathcal{I}}}(\mu)$. □

Theorem 6. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space and $\lambda \in I^X$. Then:*

(i) $\mathbf{1} - \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) = \mathrm{Int}_{h_{\mathcal{I}}}(\mathbf{1} - \lambda)$;
(ii) $\mathbf{1} - \mathrm{Int}_{h_{\mathcal{I}}}(\lambda) = \mathrm{cl}_{h_{\mathcal{I}}}(\mathbf{1} - \lambda)$.

Proof. (i) Using the definitions:

$$\begin{aligned}
\mathbf{1} - \mathrm{cl}_{h_{\mathcal{I}}}(\lambda) &= \mathbf{1} - \bigwedge\{\nu : \nu \text{ is fuzzy } h_{\mathcal{I}}\text{-closed}, \lambda \leq \nu\} \\
&= \bigvee\{\mathbf{1} - \nu : \nu \text{ is fuzzy } h_{\mathcal{I}}\text{-closed}, \lambda \leq \nu\} \\
&= \bigvee\{\gamma \in \tilde{\tau}^{h_{\mathcal{I}}} : \gamma \leq \mathbf{1} - \lambda\} \\
&= \mathrm{Int}_{h_{\mathcal{I}}}(\mathbf{1} - \lambda),
\end{aligned}$$

where the third equality uses the bijection $\nu \mapsto \gamma = \mathbf{1} - \nu$ between fuzzy $h_{\mathcal{I}}$-closed sets containing λ and fuzzy $h_{\mathcal{I}}$-open sets contained in $\mathbf{1} - \lambda$.

(ii) Replace λ by $\mathbf{1} - \lambda$ in (i): $\mathbf{1} - \mathrm{cl}_{h_{\mathcal{I}}}(\mathbf{1} - \lambda) = \mathrm{Int}_{h_{\mathcal{I}}}(\lambda)$. Hence, $\mathrm{cl}_{h_{\mathcal{I}}}(\mathbf{1} - \lambda) = \mathbf{1} - \mathrm{Int}_{h_{\mathcal{I}}}(\lambda)$. □

6. Fuzzy $h_{\mathcal{I}}$-T_1 Separation and Fuzzy $h_{\mathcal{I}}$-Subspaces

In this section, we introduce a separation axiom and a subspace construction for the $h_{\mathcal{I}}$-topology, demonstrating that this topology supports a rich structural theory.

Definition 12. *A fuzzy ideal topological space $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ is called fuzzy $h_{\mathcal{I}}$-T_1 if, for each pair of distinct points $x, y \in X$, there exist fuzzy $h_{\mathcal{I}}$-open sets $\lambda, \mu \in \tilde{\tau}^{h_{\mathcal{I}}}$ such that $\lambda(x) > 0$, $\lambda(y) = 0$, $\mu(y) > 0$, and $\mu(x) = 0$.*

Theorem 7. *For a fuzzy ideal topological space $(X, \tilde{\tau}, \tilde{\mathcal{I}})$, the following are equivalent:*

(i) $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ *is fuzzy* $h_{\mathcal{I}}$-T_1;
(ii) *For each* $x \in X$, *the fuzzy point* $\mathbf{1}_{\{x\}}$ *(i.e., the characteristic function of* $\{x\}$*) is fuzzy* $h_{\mathcal{I}}$*-closed;*
(iii) *For each* $x \in X$, $\mathrm{cl}_{h_{\mathcal{I}}}(\mathbf{1}_{\{x\}}) = \mathbf{1}_{\{x\}}$.

Proof. (i) ⇒ (ii): Let $x \in X$. We show that $\mathbf{1} - \mathbf{1}_{\{x\}}$ is fuzzy $h_{\mathcal{I}}$-open. For each $y \neq x$, by the $h_{\mathcal{I}}$-T_1 condition, there exists $\mu_y \in \tilde{\tau}^{h_{\mathcal{I}}}$ with $\mu_y(y) > 0$ and $\mu_y(x) = 0$. Then, $\mu_y \leq \mathbf{1} - \mathbf{1}_{\{x\}}$. Hence, $\mathbf{1} - \mathbf{1}_{\{x\}} = \bigvee_{y \neq x} \mu_y \in \tilde{\tau}^{h_{\mathcal{I}}}$.

(ii) ⇒ (iii): This follows directly from Theorem 5(iv).

(iii) ⇒ (i): Let $x \neq y$. Since $\mathbf{1}_{\{x\}}$ is fuzzy $h_{\mathcal{I}}$-closed, $\lambda = \mathbf{1} - \mathbf{1}_{\{x\}} \in \tilde{\tau}^{h_{\mathcal{I}}}$ with $\lambda(y) = 1 > 0$ and $\lambda(x) = 0$ and similarly for $\mu = \mathbf{1} - \mathbf{1}_{\{y\}}$. □

Example 9. *Consider* $X = \{a, b\}$, $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, (0.5, 0)\}$, $\tilde{\mathcal{I}} = \{\mathbf{0}\}$. *If* $\tilde{\tau}^{h_{\mathcal{I}}}$ *contains both* $\mathbf{1}_{\{a\}} = (1, 0)$ *and* $\mathbf{1}_{\{b\}} = (0, 1)$ *as fuzzy* $h_{\mathcal{I}}$*-closed sets (i.e., their complements* $(0, 1)$ *and* $(1, 0)$ *are fuzzy* $h_{\mathcal{I}}$*-open), then* $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ *is fuzzy* $h_{\mathcal{I}}$-T_1.

Definition 13. *Let* $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ *be a fuzzy ideal topological space and* $A \subseteq X$ *be non-empty. The fuzzy* $h_{\mathcal{I}}$*-subspace topology on* A *is defined by*

$$\tilde{\tau}_A^{h_{\mathcal{I}}} = \{\lambda|_A : \lambda \in \tilde{\tau}^{h_{\mathcal{I}}}\},$$

where $\lambda|_A$ *denotes the restriction of* λ *to* A.

Proposition 2. *Let* $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ *be a fuzzy ideal topological space and* $A \subseteq X$. *Then,* $\tilde{\tau}_A^{h_{\mathcal{I}}}$ *is a fuzzy topology on* A.

Proof. Since $\tilde{\tau}^{h_{\mathcal{I}}}$ is a fuzzy topology on X (Theorem 2), the collection of restrictions $\{\lambda|_A : \lambda \in \tilde{\tau}^{h_{\mathcal{I}}}\}$ is a fuzzy topology on A: it contains $\mathbf{0}_A$ and $\mathbf{1}_A$ (restrictions of $\mathbf{0}$ and $\mathbf{1}$), and is closed under finite infima $((\lambda_1|_A) \wedge (\lambda_2|_A) = (\lambda_1 \wedge \lambda_2)|_A)$ and under arbitrary suprema $(\bigvee_\alpha (\lambda_\alpha|_A) = (\bigvee_\alpha \lambda_\alpha)|_A)$. □

Proposition 3. *Let* $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ *be a fuzzy ideal topological space and* $A \subseteq X$ *be non-empty. Suppose the following two conditions hold:*

(i) *Every non-trivial fuzzy open set in* $\tilde{\tau}_A = \{\mu|_A : \mu \in \tilde{\tau}\}$ *is the restriction of a non-trivial fuzzy open set in* $\tilde{\tau}$*; that is, if* $\nu \in \tilde{\tau}_A$ *with* $\mathbf{0}_A \neq \nu \neq \mathbf{1}_A$, *then there exists* $\mu \in \tilde{\tau}$ *with* $\mathbf{0} \neq \mu \neq \mathbf{1}$ *and* $\mu|_A = \nu$.
(ii) *The* $*$*-closure commutes with restriction:* $\mathrm{Cl}_A^*(\mu|_A) = \mathrm{Cl}^*(\mu)|_A$ *for all* $\mu \in \tilde{\tau}$.

Then, $\tilde{\tau}_A^{h_{\mathcal{I}}} = \tilde{\tau}_A^{h_{\mathcal{I}_A}}$*; that is, restricting the* $h_{\mathcal{I}}$*-topology to* A *yields the same result as constructing the* $h_{\mathcal{I}_A}$*-topology on the subspace.*

Proof. Let $\lambda|_A \in \tilde{\tau}_A^{h_{\mathcal{I}}}$, so $\lambda \in \tilde{\tau}^{h_{\mathcal{I}}}$. For any non-trivial $\nu \in \tilde{\tau}_A$, by condition (i), there exists a non-trivial $\mu \in \tilde{\tau}$ with $\mu|_A = \nu$. Since λ is fuzzy $h_{\mathcal{I}}$-open, $\lambda \leq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$. Restricting to A and using condition (ii):

$$\lambda|_A \leq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))|_A \leq \mathrm{int}_A(\lambda|_A \vee \mathrm{Cl}_A^*(\nu)).$$

Hence, $\lambda|_A$ is fuzzy $h_{\mathcal{I}_A}$-open in $(A, \tilde{\tau}_A, \tilde{\mathcal{I}}_A)$, giving $\tilde{\tau}_A^{h_{\mathcal{I}}} \subseteq \tilde{\tau}_A^{h_{\mathcal{I}_A}}$. The reverse inclusion follows by a symmetric argument, extending fuzzy sets on A to X via zero extension and applying the same conditions. □

Remark 7. *Condition (ii) in Proposition 3 is the key requirement. It holds automatically when $\tilde{\tau}$ is the discrete topology (since Cl^* reduces to the identity) or when A is a clopen set in $(X, \tilde{\tau})$. In contrast, for non-clopen subsets of non-discrete spaces, the $*$-closure may fail to commute with restriction, causing the two subspace constructions to differ. The question of characterizing all subsets A for which the two constructions coincide remains an interesting open problem.*

7. Fuzzy $h_{\mathcal{I}}$-Continuous Functions

Definition 14. *A function $f\colon (X, \tilde{\tau}) \to (Y, \tilde{\sigma})$ is said to be fuzzy h-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy h-open in $(X, \tilde{\tau})$.*

Definition 15. *A function $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Y, \tilde{\sigma})$ is said to be fuzzy $h_{\mathcal{I}}$-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy $h_{\mathcal{I}}$-open in $(X, \tilde{\tau}, \tilde{\mathcal{I}})$.*

Theorem 8. *Every fuzzy h-continuous function is fuzzy $h_{\mathcal{I}}$-continuous.*

Proof. This follows immediately from Theorem 1. □

The converse of Theorem 8 does not hold in general.

Example 10. *Let $X = \{p, q, r\}$, $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \chi_{\{q\}}, \chi_{\{q,r\}}\}$, $\tilde{\mathcal{I}} = \{\lambda \in I^X : \mathrm{supp}(\lambda) \subseteq \{p\}\}$, and $\tilde{\sigma} = \{\mathbf{0}, \mathbf{1}, \chi_{\{p,q\}}\}$. Let $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (X, \tilde{\sigma})$ be the identity function. Then, f is fuzzy $h_{\mathcal{I}}$-continuous but not fuzzy h-continuous, since $f^{-1}(\chi_{\{p,q\}}) = \chi_{\{p,q\}} \in \tilde{\tau}^{h_{\mathcal{I}}}$ but $\chi_{\{p,q\}} \notin \tilde{\tau}^h$.*

Definition 16. *A function $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Y, \tilde{\sigma})$ is said to be:*

(i) Fuzzy $h\alpha_{\mathcal{I}}$-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy $h\alpha_{\mathcal{I}}$-open in X;
(ii) Fuzzy $hp_{\mathcal{I}}$-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy $hp_{\mathcal{I}}$-open in X;
(iii) Fuzzy $hs_{\mathcal{I}}$-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy $hs_{\mathcal{I}}$-open in X;
(iv) Fuzzy $h\beta_{\mathcal{I}}$-continuous if, for each $\nu \in \tilde{\sigma}$, $f^{-1}(\nu)$ is fuzzy $h\beta_{\mathcal{I}}$-open in X.

Corollary 1. *A function $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Y, \tilde{\sigma})$ is fuzzy $h_{\mathcal{I}}$-continuous if and only if $f\colon (X, \tilde{\tau}^{h_{\mathcal{I}}}) \to (Y, \tilde{\sigma})$ is fuzzy continuous.*

Proof. This is an immediate consequence of Theorem 2. □

Theorem 9. *If $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Y, \tilde{\sigma})$ is fuzzy $h_{\mathcal{I}}$-continuous and $g\colon (Y, \tilde{\sigma}) \to (Z, \tilde{\gamma})$ is fuzzy continuous, then $g \circ f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Z, \tilde{\gamma})$ is fuzzy $h_{\mathcal{I}}$-continuous.*

Proof. For any $\nu \in \tilde{\gamma}$, $g^{-1}(\nu) \in \tilde{\sigma}$, since g is fuzzy continuous. Then, $(g \circ f)^{-1}(\nu) = f^{-1}(g^{-1}(\nu))$ is fuzzy $h_{\mathcal{I}}$-open in $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ since f is fuzzy $h_{\mathcal{I}}$-continuous. □

Lemma 1. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space. A fuzzy set $\beta \in I^X$ is fuzzy $h_{\mathcal{I}}$-closed if and only if $\mathrm{cl}(\beta \wedge \mathrm{Int}^*(\varphi)) \leq \beta$ for all fuzzy closed sets φ in X such that $\mathbf{0} \neq \varphi \neq \mathbf{1}$.*

Proof. β is fuzzy $h_{\mathcal{I}}$-closed if and only if $\mathbf{1} - \beta$ is fuzzy $h_{\mathcal{I}}$-open, i.e., $\mathbf{1} - \beta \leq \mathrm{int}((\mathbf{1} - \beta) \vee \mathrm{Cl}^*(\mu))$ for all $\mu \in \tilde{\tau}$ with $\mathbf{0} \neq \mu \neq \mathbf{1}$. This is equivalent to

$$\mathbf{1} - \mathrm{int}((\mathbf{1} - \beta) \vee \mathrm{Cl}^*(\mu)) \leq \beta.$$

Now,

https://doi.org/10.3390/math14050904

$$\begin{aligned}\mathbf{1}-\mathrm{int}((\mathbf{1}-\beta)\vee \mathrm{Cl}^*(\mu)) &= \mathrm{cl}(\mathbf{1}-((\mathbf{1}-\beta)\vee \mathrm{Cl}^*(\mu)))\\ &= \mathrm{cl}(\beta\wedge(\mathbf{1}-\mathrm{Cl}^*(\mu)))\\ &= \mathrm{cl}(\beta\wedge \mathrm{Int}^*(\mathbf{1}-\mu)).\end{aligned}$$

Setting $\varphi = \mathbf{1}-\mu$ (which is fuzzy closed when μ is fuzzy open), we obtain $\mathrm{cl}(\beta\wedge\mathrm{Int}^*(\varphi)) \le \beta$ for all fuzzy closed φ with $\mathbf{0}\ne\varphi\ne\mathbf{1}$. □

Theorem 10. *For a function $f\colon (X,\tilde{\tau},\tilde{\mathcal{I}}) \to (Y,\tilde{\sigma})$, the following properties are equivalent:*

(i) f is fuzzy $h_{\mathcal{I}}$-continuous;

(ii) For each $x\in X$ and each fuzzy open ν in Y with $\nu(f(x))>0$, there exists $\gamma\in\tilde{\tau}^{h_{\mathcal{I}}}$ with $\gamma(x)>0$ and $f(\gamma)\le\nu$;

(iii) For each $x\in X$ and each fuzzy open set ν of Y with $\nu(f(x))>0$, there exists a fuzzy $h_{\mathcal{I}}$-neighborhood ω of x such that $f(\omega)\le\nu$;

(iv) The inverse image of each fuzzy closed set in Y is fuzzy $h_{\mathcal{I}}$-closed in X;

(v) For each fuzzy set $\beta\in I^Y$, $\mathrm{cl}_{h_{\mathcal{I}}}(f^{-1}(\beta))\le f^{-1}(\mathrm{cl}(\beta))$;

(vi) For each fuzzy set $\beta\in I^Y$, $f^{-1}(\mathrm{int}(\beta))\le \mathrm{Int}_{h_{\mathcal{I}}}(f^{-1}(\beta))$.

Proof. (i) ⇒ (ii): Let $\nu\in\tilde{\sigma}$ with $\nu(f(x))>0$. Set $\gamma=f^{-1}(\nu)$. Then, γ is fuzzy $h_{\mathcal{I}}$-open by (i), $\gamma(x)=\nu(f(x))>0$, and $f(\gamma)\le\nu$.

(ii) ⇒ (iii): Every fuzzy $h_{\mathcal{I}}$-open set with positive value at x is a fuzzy $h_{\mathcal{I}}$-neighborhood of x.

(iii) ⇒ (i): Let $\nu\in\tilde{\sigma}$. For each x with $f^{-1}(\nu)(x)>0$, by (iii), there exists a fuzzy $h_{\mathcal{I}}$-neighborhood ω_x of x with $f(\omega_x)\le\nu$; therefore, $\omega_x\le f^{-1}(\nu)$. There exists $\gamma_x\in\tilde{\tau}^{h_{\mathcal{I}}}$ with $\gamma_x(x)>0$ and $\gamma_x\le\omega_x\le f^{-1}(\nu)$. Hence, $f^{-1}(\nu)=\bigvee_x\gamma_x\in\tilde{\tau}^{h_{\mathcal{I}}}$.

(i) ⇔ (iv): Clear, since $f^{-1}(\mathbf{1}-\nu)=\mathbf{1}-f^{-1}(\nu)$.

(iv) ⇒ (v): For $\beta\in I^Y$, $f^{-1}(\mathrm{cl}(\beta))$ is fuzzy $h_{\mathcal{I}}$-closed and $f^{-1}(\beta)\le f^{-1}(\mathrm{cl}(\beta))$. Therefore, $\mathrm{cl}_{h_{\mathcal{I}}}(f^{-1}(\beta))\le f^{-1}(\mathrm{cl}(\beta))$.

(v) ⇒ (vi): For $\beta\in I^Y$,

$$\begin{aligned}f^{-1}(\mathrm{int}(\beta)) &= f^{-1}(\mathbf{1}-\mathrm{cl}(\mathbf{1}-\beta))\\ &= \mathbf{1}-f^{-1}(\mathrm{cl}(\mathbf{1}-\beta))\\ &\le \mathbf{1}-\mathrm{cl}_{h_{\mathcal{I}}}(f^{-1}(\mathbf{1}-\beta))\\ &= \mathbf{1}-\mathrm{cl}_{h_{\mathcal{I}}}(\mathbf{1}-f^{-1}(\beta))\\ &= \mathrm{Int}_{h_{\mathcal{I}}}(f^{-1}(\beta)).\end{aligned}$$

(vi) ⇒ (i): For $\nu\in\tilde{\sigma}$, by (vi), $f^{-1}(\nu)\le\mathrm{Int}_{h_{\mathcal{I}}}(f^{-1}(\nu))\le f^{-1}(\nu)$. Hence, $\mathrm{Int}_{h_{\mathcal{I}}}(f^{-1}(\nu))=f^{-1}(\nu)$, so $f^{-1}(\nu)$ is fuzzy $h_{\mathcal{I}}$-open. □

8. Fuzzy $h_{\mathcal{I}}$-Irresolute and Fuzzy $h_{\mathcal{I}}$-Open Functions

Definition 17. *A function $f\colon (X,\tilde{\tau},\tilde{\mathcal{I}})\to(Y,\tilde{\sigma},\tilde{\mathcal{J}})$ is said to be fuzzy $h_{\mathcal{I}}$-irresolute if, for each fuzzy $h_{\mathcal{J}}$-open set ν in Y, $f^{-1}(\nu)$ is fuzzy $h_{\mathcal{I}}$-open in X.*

Theorem 11. *If a function $f\colon (X,\tilde{\tau},\tilde{\mathcal{I}})\to(Y,\tilde{\sigma},\tilde{\mathcal{J}})$ is fuzzy $h_{\mathcal{I}}$-irresolute, then f is fuzzy $h_{\mathcal{I}}$-continuous.*

Proof. Let $\nu\in\tilde{\sigma}$. Since every fuzzy open set is fuzzy h-open and every fuzzy h-open set is fuzzy $h_{\mathcal{J}}$-open (by Theorem 1), ν is fuzzy $h_{\mathcal{J}}$-open. By fuzzy $h_{\mathcal{I}}$-irresoluteness, $f^{-1}(\nu)$ is fuzzy $h_{\mathcal{I}}$-open. Hence, f is fuzzy $h_{\mathcal{I}}$-continuous. □

The converse of Theorem 11 does not hold in general.

Example 11. *Let $X = \{p, q, r, s\}$, $\tilde{\tau} = \{\mathbf{0}, \mathbf{1}, \chi_{\{q\}}, \chi_{\{r,s\}}, \chi_{\{q,r,s\}}\}$, $\tilde{\mathcal{I}} = \{\lambda \in I^X : \mathrm{supp}(\lambda) \subseteq \{q\}\}$, $Y = \{p, q, r\}$, $\tilde{\sigma} = \{\mathbf{0}, \mathbf{1}_Y, \chi_{\{p\}}, \chi_{\{q,r\}}\}$, and $\tilde{\mathcal{J}} = \{\lambda \in I^Y : \mathrm{supp}(\lambda) \subseteq \{p\}\}$. Define $f\colon X \to Y$ by $f(p) = q$, $f(q) = p$, $f(r) = r$, $f(s) = q$. Then, f is fuzzy $h_{\mathcal{I}}$-continuous but not fuzzy $h_{\mathcal{I}}$-irresolute, since, for $\chi_{\{p,r\}} \in \tilde{\sigma}^{h_{\mathcal{J}}}$, we have $f^{-1}(\chi_{\{p,r\}}) = \chi_{\{q,r\}} \notin \tilde{\tau}^{h_{\mathcal{I}}}$.*

Remark 8. *Fuzzy $h_{\mathcal{I}}$-irresolute functions are not necessarily fuzzy continuous, and fuzzy continuous functions are not necessarily fuzzy $h_{\mathcal{I}}$-irresolute. Thus, fuzzy continuity and fuzzy $h_{\mathcal{I}}$-irresoluteness are independent concepts.*

Corollary 2. *A function $f\colon (X, \tilde{\tau}, \tilde{\mathcal{I}}) \to (Y, \tilde{\sigma}, \tilde{\mathcal{J}})$ is fuzzy $h_{\mathcal{I}}$-irresolute if and only if $f\colon (X, \tilde{\tau}^{h_{\mathcal{I}}}) \to (Y, \tilde{\sigma}^{h_{\mathcal{J}}})$ is fuzzy continuous.*

Proof. This is an immediate consequence of Theorem 2. □

Definition 18. *A function $f\colon (X, \tilde{\tau}) \to (Y, \tilde{\sigma}, \tilde{\mathcal{J}})$ is said to be fuzzy $h_{\mathcal{I}}$-open if $f(\nu)$ is fuzzy $h_{\mathcal{J}}$-open in Y for every $\nu \in \tilde{\tau}$.*

Proposition 4. *Every fuzzy open function is fuzzy $h_{\mathcal{I}}$-open.*

Proof. Since every fuzzy open set is fuzzy h-open and every fuzzy h-open set is fuzzy $h_{\mathcal{J}}$-open (by Theorem 1), if f is fuzzy open, then $f(\nu) \in \tilde{\sigma}$ for all $\nu \in \tilde{\tau}$, whence $f(\nu)$ is fuzzy $h_{\mathcal{J}}$-open. □

Theorem 12. *A function $f\colon (X, \tilde{\tau}) \to (Y, \tilde{\sigma}, \tilde{\mathcal{J}})$ is fuzzy $h_{\mathcal{I}}$-open if and only if, for each fuzzy set $\omega \in I^Y$ and each fuzzy closed set φ of X with $f^{-1}(\omega) \leq \varphi$, there exists a fuzzy $h_{\mathcal{J}}$-closed set ψ in Y with $\omega \leq \psi$ and $f^{-1}(\psi) \leq \varphi$.*

Proof. *Necessity.* Let $\psi = \mathbf{1} - f(\mathbf{1} - \varphi)$. Since $f^{-1}(\omega) \leq \varphi$, we have $f(\mathbf{1} - \varphi) \leq \mathbf{1} - \omega$. Since f is fuzzy $h_{\mathcal{I}}$-open, $f(\mathbf{1} - \varphi)$ is fuzzy $h_{\mathcal{J}}$-open (as $\mathbf{1} - \varphi$ is fuzzy open); therefore, ψ is fuzzy $h_{\mathcal{J}}$-closed. Also, $\omega \leq \psi$ and $f^{-1}(\psi) = \mathbf{1} - f^{-1}(f(\mathbf{1} - \varphi)) \leq \mathbf{1} - (\mathbf{1} - \varphi) = \varphi$.

Sufficiency. Let $\nu \in \tilde{\tau}$ and set $\omega = \mathbf{1} - f(\nu)$. Then, $f^{-1}(\omega) = \mathbf{1} - f^{-1}(f(\nu)) \leq \mathbf{1} - \nu$, and $\mathbf{1} - \nu$ is fuzzy closed. By hypothesis, there exists a fuzzy $h_{\mathcal{J}}$-closed set ψ with $\omega \leq \psi$ and $f^{-1}(\psi) \leq \mathbf{1} - \nu$. Then, $\psi \leq \mathbf{1} - f(\nu)$. Therefore, $\mathbf{1} - f(\nu) \leq \psi \leq \mathbf{1} - f(\nu)$, so $f(\nu) = \mathbf{1} - \psi$ is fuzzy $h_{\mathcal{J}}$-open. □

Proposition 5. *If $f\colon (X, \tilde{\tau}) \to (Y, \tilde{\sigma})$ is fuzzy open and $g\colon (Y, \tilde{\sigma}) \to (Z, \tilde{\gamma}, \tilde{\mathcal{J}})$ is fuzzy $h_{\mathcal{J}}$-open, then $g \circ f\colon (X, \tilde{\tau}) \to (Z, \tilde{\gamma}, \tilde{\mathcal{J}})$ is fuzzy $h_{\mathcal{J}}$-open.*

Proof. For $\nu \in \tilde{\tau}$, $f(\nu) \in \tilde{\sigma}$ (since f is fuzzy open), and $g(f(\nu))$ is fuzzy $h_{\mathcal{J}}$-open in Z (since g is fuzzy $h_{\mathcal{J}}$-open). Hence, $(g \circ f)(\nu) = g(f(\nu))$ is fuzzy $h_{\mathcal{J}}$-open. □

9. Application to Multi-Criteria Decision Making

In this section, we demonstrate a concrete application of the fuzzy $h_{\mathcal{I}}$-interior to a multi-criteria decision-making (MCDM) problem, showing how the ideal filters negligible criteria and the $h_{\mathcal{I}}$-topology produces a refined ranking.

9.1. Problem Formulation

Consider an MCDM problem with a set of alternatives $X = \{a_1, a_2, a_3, a_4\}$ and criteria $C_1, \ldots, C_5$ (e.g., cost, quality, delivery time, reliability, and environmental impact). Each alternative is evaluated by a fuzzy set $\mu_j \in I^X$ representing performance under criterion C_j, where $\mu_j(a_i) \in [0, 1]$ denotes the degree to which alternative a_i satisfies criterion C_j.

Suppose the performance evaluations are

	C_1	C_2	C_3	C_4	C_5
a_1	0.8	0.6	0.7	0.9	0.3
a_2	0.5	0.9	0.4	0.6	0.8
a_3	0.7	0.7	0.8	0.5	0.6
a_4	0.6	0.5	0.9	0.7	0.4

9.2. Fuzzy Topology and Ideal Construction

We construct the fuzzy topology $\tilde{\tau}$ by taking the initial topology generated by $\{\mu_1, \ldots, \mu_5\}$ (including all finite meets, arbitrary joins, $\mathbf{0}$, and $\mathbf{1}$). The criteria-generated opens include μ_j and their pairwise meets $\mu_j \wedge \mu_k$.

The decision maker identifies criteria with low discriminative power as negligible. Define the fuzzy ideal

$$\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a_i) \leq 0.35 \text{ for all } i\}.$$

This ideal captures fuzzy sets that assign at most marginal membership to all alternatives—intuitively, criteria or sub-criteria that fail to meaningfully distinguish between alternatives.

9.3. Computing $\tilde{\tau}^{h_{\mathcal{I}}}$ and the Refined Ranking

Step 1: Aggregate score. A natural decision fuzzy set is the weighted average $\lambda^{\text{agg}} = \frac{1}{5}(\mu_1 \vee \mu_2 \vee \mu_3 \vee \mu_4 \vee \mu_5)$, representing the "best criterion" performance per alternative:

$$\lambda^{\text{agg}} = (0.9, 0.9, 0.8, 0.9).$$

Under $\tilde{\tau}$ alone, all alternatives except a_3 appear tied, making it difficult to rank them.

Step 2: Standard fuzzy interior. The fuzzy interior $\text{int}(\lambda^{\text{agg}})$ returns the largest fuzzy open set below λ^{agg}. Since λ^{agg} may not itself be open, $\text{int}(\lambda^{\text{agg}})$ may lose significant information.

Step 3: Fuzzy $h_{\mathcal{I}}$-interior. We compute $\text{Int}_{h_{\mathcal{I}}}(\lambda^{\text{agg}})$ using the $h_{\mathcal{I}}$-topology. By the definition, λ^{agg} is fuzzy $h_{\mathcal{I}}$-open if $\lambda^{\text{agg}} \leq \text{int}(\lambda^{\text{agg}} \vee \text{Cl}^*(\mu))$ for all non-trivial $\mu \in \tilde{\tau}$. Since $\text{Cl}^*(\mu) \geq \mu$ typically inflates each criterion's evaluation, the condition becomes easier to satisfy, and more fuzzy sets qualify as $h_{\mathcal{I}}$-open.

Step 4: Ideal-refined aggregation. Instead of using the maximum, we define an ideal-sensitive aggregation:

$$\lambda^{\text{ref}}(a_i) = \frac{1}{|\mathcal{C}_i|} \sum_{j \in \mathcal{C}_i} \mu_j(a_i), \quad \text{where } \mathcal{C}_i = \{j : \mu_j(a_i) > 0.35\}.$$

This counts only criteria exceeding the ideal threshold for each alternative:

	C_1	C_2	C_3	C_4	C_5	λ^{ref}
a_1	0.8	0.6	0.7	0.9	–	0.750
a_2	0.5	0.9	0.4	0.6	0.8	0.640
a_3	0.7	0.7	0.8	0.5	0.6	0.660
a_4	0.6	0.5	0.9	0.7	0.4	0.620

For a_1, C_5 is filtered (membership $0.3 \leq 0.35$) and the remaining four strong criteria yield an average of 0.75. For a_2, all five criteria contribute. The resulting ranking is $a_1 \succ a_3 \succ a_2 \succ a_4$.

9.4. Comparison of Rankings

The $h_{\mathcal{I}}$-refined method produces a *complete* ranking where all alternatives are distinguishable, whereas both the max-aggregation and the arithmetic mean produce ties. The key advantage is that the ideal filters out marginally-performing criteria for each alternative, preventing low scores from diluting strong performers (the arithmetic mean problem) while also preventing a single high score from masking overall mediocrity (the max-aggregation problem) (Table 2).

Table 2. Comparison of aggregation methods: scores, rankings, and distinguishability of alternatives.

Method	Score	Ranking	Distinguishable?
max-aggregation ($\tilde{\tau}$)	$(0.9, 0.9, 0.8, 0.9)$	$a_1 \sim a_2 \sim a_4 \succ a_3$	Partial
Arithmetic mean	$(0.66, 0.64, 0.66, 0.62)$	$a_1 \sim a_3 \succ a_2 \succ a_4$	Partial
$h_{\mathcal{I}}$-refined ($\tilde{\tau}^{h_{\mathcal{I}}}$)	$(0.75, 0.64, 0.66, 0.62)$	$a_1 \succ a_3 \succ a_2 \succ a_4$	Full

9.5. Theoretical Justification

The connection to our theory is as follows: The ideal $\tilde{\mathcal{I}}$ filters negligible criteria, and the Cl^* operator inflates the remaining criteria to reflect their "true importance" after ideal correction. The $h_{\mathcal{I}}$-openness condition $\lambda \leq \mathrm{int}(\lambda \vee \mathrm{Cl}^*(\mu))$ ensures that the aggregated score is robust with respect to perturbations by any criterion (including ideal-corrected ones). The $h_{\mathcal{I}}$-interior $\mathrm{Int}_{h_{\mathcal{I}}}(\lambda^{\mathrm{ref}})$ then provides the largest "stable" scoring function below the raw scores—the one that is preserved under all ideal-corrected perturbations.

This example demonstrates that the $h_{\mathcal{I}}$-topology provides a principled mathematical framework for handling the ubiquitous problem of negligible or dominated criteria in MCDM, going beyond ad hoc threshold methods by leveraging the algebraic structure of fuzzy ideals and the topological structure of h-openness.

9.6. Necessity of Ideal Filtering

Proposition 6. *Let $(X, \tilde{\tau}, \tilde{\mathcal{I}})$ be a fuzzy ideal topological space with the MCDM setup of Section 9. If $\tilde{\mathcal{I}} = I^X$ (the maximal ideal, i.e., no criteria are filtered), then $\tilde{\tau}^{h_{\mathcal{I}}} = \tilde{\tau}^h$ and the refined ranking reduces to the standard h-open ranking, which may produce ties. Conversely, if $\tilde{\mathcal{I}}$ is a proper non-trivial ideal (some criteria are filtered), then $\tilde{\tau}^h \subseteq \tilde{\tau}^{h_{\mathcal{I}}}$ with the inclusion being potentially strict, and the refined ranking may break ties that the standard ranking cannot resolve.*

Proof. The first statement follows directly from Proposition 3.5: when $\tilde{\mathcal{I}} = I^X$, $\lambda^* = \mathbf{0}$ for all λ, so $\mathrm{Cl}^*(\mu) = \mu$ and the $h_{\mathcal{I}}$-openness condition reduces to h-openness. For the second statement, when $\tilde{\mathcal{I}}$ is proper, there exists μ with $\mathrm{Cl}^*(\mu) > \mu$. Then, $\lambda \vee \mathrm{Cl}^*(\mu) \geq \lambda \vee \mu$, with equality not always holding, so the $h_{\mathcal{I}}$-openness condition is strictly weaker than h-openness, yielding $\tilde{\tau}^h \subseteq \tilde{\tau}^{h_{\mathcal{I}}}$ with potentially strict inclusion. In the MCDM context, this means more fuzzy sets qualify as $h_{\mathcal{I}}$-open, allowing $\mathrm{Int}_{h_{\mathcal{I}}}$ to preserve more information than Int_h, thereby breaking ties. □

9.7. Sensitivity Analysis

To demonstrate the robustness of the $h_{\mathcal{I}}$-refined ranking, we vary the ideal threshold parameter δ (where $\tilde{\mathcal{I}} = \{\lambda \in I^X : \lambda(a_i) \leq \delta \text{ for all } i\}$) and observe its effect on the ranking.

Table 3 shows that the ranking is stable for $\delta \in [0.30, 0.35]$, producing the complete ordering $a_1 \succ a_3 \succ a_2 \succ a_4$. For smaller thresholds ($\delta \leq 0.25$), fewer criteria are filtered, and a tie between a_1 and a_3 emerges. For larger thresholds ($\delta = 0.40$), over-filtering occurs: the ranking changes to $a_1 \succ a_2 \succ a_4 \succ a_3$, as the removal of marginally-performing criteria alters the relative strengths. This confirms that the ideal parameter δ provides meaningful control over the filtering granularity, and that a moderate threshold yields the most informative ranking.

Table 3. Sensitivity of the refined ranking to the ideal threshold δ.

δ	$\lambda^{\text{ref}}(a_1)$	$\lambda^{\text{ref}}(a_2)$	$\lambda^{\text{ref}}(a_3)$	$\lambda^{\text{ref}}(a_4)$	**Ranking**
0.20	0.660	0.640	0.660	0.620	$a_1 \sim a_3 \succ a_2 \succ a_4$
0.25	0.660	0.640	0.660	0.620	$a_1 \sim a_3 \succ a_2 \succ a_4$
0.30	0.750	0.640	0.660	0.620	$a_1 \succ a_3 \succ a_2 \succ a_4$
0.35	0.750	0.640	0.660	0.620	$a_1 \succ a_3 \succ a_2 \succ a_4$
0.40	0.750	0.700	0.660	0.675	$a_1 \succ a_2 \succ a_4 \succ a_3$

9.8. Comparison of Methods

Table 4 demonstrates that only the $h_{\mathcal{I}}$-refined method produces a *complete* ranking with all four alternatives fully distinguishable. The max-aggregation suffers from a three-way tie, and both the arithmetic mean and the standard h-open ranking (which corresponds to $\tilde{\mathcal{I}} = I^X$, i.e., no ideal filtering, by Proposition 6) produce a two-way tie between a_1 and a_3. The $h_{\mathcal{I}}$-method resolves all ties by filtering the negligible criterion (C_5 for a_1, with score $0.3 \leq 0.35$) through the ideal before aggregation, allowing a_1's strong performance on the remaining four criteria to be properly reflected.

Table 4. Detailed numerical comparison of aggregation methods.

Method	a_1	a_2	a_3	a_4	**Ranking**
Max-aggregation	0.900	0.900	0.800	0.900	$a_1 \sim a_2 \sim a_4 \succ a_3$
Arithmetic mean	0.660	0.640	0.660	0.620	$a_1 \sim a_3 \succ a_2 \succ a_4$
h-open ($\tilde{\mathcal{I}} = I^X$)	0.660	0.640	0.660	0.620	$a_1 \sim a_3 \succ a_2 \succ a_4$
$\text{Int}_{h_{\mathcal{I}}}$ ($\delta = 0.35$)	0.750	0.640	0.660	0.620	$a_1 \succ a_3 \succ a_2 \succ a_4$

9.9. Connection to the Interior Operator Properties

Remark 9. *The stability of the refined ranking is a direct consequence of the idempotency property* $\text{Int}_{h_{\mathcal{I}}}(\text{Int}_{h_{\mathcal{I}}}(\lambda)) = \text{Int}_{h_{\mathcal{I}}}(\lambda)$ *(Theorem 4(iii)). This ensures that applying the $h_{\mathcal{I}}$-interior to the refined scores λ^{ref} produces a fixed point. Repeated application of the filtering procedure does not change the ranking. In contrast, ad hoc threshold methods without topological grounding may produce unstable rankings that change with each iteration.*

10. Conclusions and Future Work

In this paper, we have introduced the notion of fuzzy $h_{\mathcal{I}}$-open sets in fuzzy ideal topological spaces $(X, \tilde{\tau}, \tilde{\mathcal{I}})$. Our main contributions are:

1. We proved that the collection of all fuzzy $h_{\mathcal{I}}$-open sets forms a fuzzy topology $\tilde{\tau}^{h_{\mathcal{I}}}$ (Theorem 2) and established that $\tilde{\tau}^*$ and $\tilde{\tau}^{h_{\mathcal{I}}}$ are in general incomparable (Theorem 3), showing that the $h_{\mathcal{I}}$-construction captures fundamentally different information from the $*$-topology.
2. We introduced and studied the hierarchy of generalized fuzzy open sets: $h\alpha_{\mathcal{I}}$-open, $hp_{\mathcal{I}}$-open, $hs_{\mathcal{I}}$-open, and $h\beta_{\mathcal{I}}$-open sets, providing genuinely fuzzy examples that show that these classes are distinct and that the size of the ideal controls the fineness of the hierarchy.
3. We developed the fuzzy $h_{\mathcal{I}}$-interior and fuzzy $h_{\mathcal{I}}$-closure operators with detailed proofs of their fundamental properties.
4. We introduced a fuzzy $h_{\mathcal{I}}$-T_1 separation axiom and a subspace construction, demonstrating that $\tilde{\tau}^{h_{\mathcal{I}}}$ supports a rich topological theory. We provided a sufficient condition (Proposition 3) under which the two natural subspace constructions coincide.

5. We introduced fuzzy $h_{\mathcal{I}}$-continuous and fuzzy $h_{\mathcal{I}}$-irresolute functions, providing comprehensive characterizations (Theorem 10).
6. We demonstrated a concrete application to multi-criteria decision making, where the ideal-based filtering produces a complete ranking that outperforms standard aggregation methods. We proved that ideal filtering is essential for tie-breaking (Proposition 6), validated the ranking's robustness through sensitivity analysis (Table 3), and connected the ranking stability to the idempotency property of the $h_{\mathcal{I}}$-interior operator.

The following open problems are suggested for future research:

1. Can new types of fuzzy h-open sets be obtained by using the fuzzy local function μ^* instead of $\mathrm{Cl}^*(\mu)$ in Definition 6?
2. Can the fuzzy $*$-interior Int^* be used instead of int (resp. αint, $p$int, $s$int, βint) to define further generalizations?
3. What are the categorical properties of the assignment $(X, \tilde{\tau}, \tilde{\mathcal{I}}) \mapsto (X, \tilde{\tau}^{h_{\mathcal{I}}})$?
4. Under what conditions on $A \subseteq X$ do we have $\tilde{\tau}_A^{h_{\mathcal{I}}} = \tilde{\tau}_A^{h_{\mathcal{I}_A}}$? Proposition 3 provides a sufficient condition; a full characterization remains open.
5. Can the $h_{\mathcal{I}}$-framework be extended to intuitionistic fuzzy or neutrosophic settings, and what additional applications arise in these generalized contexts?
6. Can the MCDM application be extended to incorporate weighted criteria and group decision making within the $h_{\mathcal{I}}$-topological framework?

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inform. Control* **1965**, *8*, 338–353. [CrossRef]
2. Bloch, I. Fuzzy sets for image processing and understanding. *Fuzzy Sets Syst.* **2015**, *281*, 280–291. [CrossRef]
3. Olgun, M.; Unver, M.; Yardimci, Ş. Pythagorean fuzzy topological spaces. *Complex Intell. Syst.* **2019**, *5*, 177–183. [CrossRef]
4. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–190. [CrossRef]
5. Azad, K.K. On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity. *J. Math. Anal. Appl.* **1981**, *82*, 14–32. [CrossRef]
6. Bin Shahna, A.S. On fuzzy strong semicontinuity and fuzzy precontinuity. *Fuzzy Sets Syst.* **1991**, *44*, 303–308. [CrossRef]
7. Rosenfeld, A. Fuzzy digital topology. *Inform. Control* **1979**, *40*, 76–87. [CrossRef]
8. Saha, P.K.; Udupa, J.K. Fuzzy digital topology and geometry and their applications to medical imaging. In *Computational Fuzzy Sets and Their Applications*; Springer: Berlin/Heidelberg, Germany, 2014; pp. 15–39.
9. Moosavi, N.; Bagheri, M.; Nabi-Bidhendi, M. Hydrocarbon reservoir parameter estimation using a fuzzy Gaussian based SVR method. *Bull. Geoph. Ocean.* **2024**, *65*, 701–714.
10. Papamarkou, T.; Birdal, T.; Bronstein, M.M.; Carlsson, G.E.; Curry, J.; Gao, Y.; Hajij, M.; Kwitt, R.; Lio, P.; Di Lorenzo, P.; et al. Position: Topological deep learning is the new frontier for relational learning. In Proceedings of the International Conference on Machine Learning (ICML), Vienna, Austria, 21–27 July 2024.
11. Zia, A.; Rasheed, A.; Azam, A.; Rashid, T. Topological deep learning: A review of an emerging paradigm. *Artif. Intell. Rev.* **2024**, *57*, 77. [CrossRef]
12. El-Gayar, M.A.; Abu-Gdairi, R. Extension of topological structures using lattices and rough sets. *AIMS Math.* **2024**, *9*, 7552–7569. [CrossRef]
13. Kumar, V.; Tiwari, S. A survey on topological structures on fuzzy rough sets. *Afr. Mat.* **2024**, *35*, 42. [CrossRef]
14. Kuratowski, K. *Topology*; Academic Press: New York, NY, USA, 1966; Volume I.
15. Vaidyanathaswamy, R. *Set Topology*; Chelsea Publishing Company: New York, NY, USA, 1960.
16. Janković, D.; Hamlett, T.R. New topologies from old via ideals. *Am. Math. Monthly* **1990**, *97*, 295–310. [CrossRef]
17. Sarkar, D. Fuzzy ideal theory, fuzzy local function and generated fuzzy topology. *Fuzzy Sets Syst.* **1997**, *87*, 117–123. [CrossRef]
18. Huang, Z.; Li, J. Covering based multi-granulation rough fuzzy sets with applications to feature selection. *Expert Syst. Appl.* **2024**, *238*, 121908. [CrossRef]

19. Zhang, P.; Li, T.; Wang, G.; Luo, C.; Chen, H.; Zhang, J.; Wang, D.; Yu, Z. Multi-source information fusion based on rough set theory: A review. *Inform. Fusion* **2021**, *68*, 85–117. [CrossRef]
20. Abbas, F.H. On h-open sets and h-continuous functions. *Bol. Soc. Paran. Mat.* **2023**, *41*, 1–14. [CrossRef]
21. Acikgoz, A.; Noiri, T. Idealizability of some expansions of open sets. *Mathematica* **2025**, *67*, 3–12. [CrossRef]
22. Abd El-Monsef, M.E.; El-Deeb, S.N.; Mahmoud, R.A. β-open sets and β-continuous mappings. *Bull. Fac. Sci. Assiut Univ.* **1983**, *12*, 77–90.