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Research paper

Enhancing decision-making with q-complex Diophantine neutrosophic normal interval-valued sets for industrial robot selection

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ABSTRACT

Robots will increasingly be able to perform everyday tasks with ease, provided they are equipped with decision-making and planning capabilities that enable them to understand and execute these tasks effectively. This research applies a multiple-attribute decision-making (MADM) approach to solve complex decision-making problems, utilizing degrees of truth functions to manage data uncertainty. The proposed q-complex Diophantine neutrosophic normal interval-valued set (q-CDNNIVS) presents a comprehensive framework for managing uncertainty, especially in cases involving periodicity. The concept of q-CDNNIVS is introduced, which uses three independent truth degrees to evaluate information. We propose several averaging and geometric aggregation operators (AOs) and discuss their algebraic properties, such as distributivity, idempotency, and associativity. A MADM algorithm is designed to address decision-making challenges, and its applicability is demonstrated through a case study on industrial robot selection that considers factors such as reach, payload, flexibility, speed, and standardization. A comparative study with existing methods highlights the conservative nature of the proposed algorithm, where sensitivity analysis and validation further support its effectiveness. The results offer decision-makers a reliable tool for managing complex, inconsistent data in industrial applications. Additionally, the importance of the parameter q is highlighted, with a comparison analysis confirming the method's feasibility and practical application.

1. Introduction

Industrial robots are programmable systems designed to perform a variety of automated tasks in manufacturing and production environments, significantly enhancing productivity, precision, and safety (Slamani et al., 2026). These robots are distinguished by their ability to execute complex movements across multiple axes with high accuracy, making them essential for industries that require repetitive, intricate, or hazardous operations. The swift advancement of industrial robotics has been propelled by innovations in artificial intelligence (AI), machine learning, sensing technologies, and human-machine collaboration, enabling the transformation of traditional automation into adaptive, intelligent systems capable of addressing diverse real-world challenges (Langas et al., 2026). This progress has empowered robots to evolve beyond fixed, task-specific functions, allowing them to adopt more flexible roles, thereby reducing cycle times and enhancing overall operational efficiency across industries such as automotive, electronics, and logistics (Choi et al., 2025).

In addition to optimizing core manufacturing processes, industrial robots contribute to broader industrial goals, including welding (Zheng et al., 2022), energy efficiency (Lu et al., 2024), manufacturing systems (Zheng et al., 2023), and dynamic control (Wang et al., 2022), etc. Collaborative robots (cobots) are specifically designed to operate safely in close proximity to human workers, eliminating the need for extensive safety barriers. This capability makes them especially advantageous in flexible production environments and small-batch manufacturing, where flexibility is paramount (Simone et al., 2022). Recent applications have demonstrated that cobots can improve assembly throughput and facilitate

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Abbreviations

MADM	Multiple attribute decision-making
AO	Aggregating operator
TD	Truth degree
ID	Indeterminacy degree
FD	False degree
FS	Fuzzy set
IFS	Intuitionistic FS
PFS	Pythagorean FS
IVPFS	Interval-valued PFS
NS	Neutrosophic set
PNS	Pythagorean NS
PNIVS	Pythagorean neutrosophic interval-valued set
CFS	Complex FS
CPFS	Complex Pythagorean FS
CIFS	Complex Intuitionistic FS
LDFS	Linear Diophantine fuzzy set
CDNNIVS	Complex Diophantine neutrosophic normal interval-valued set
q-ROFS	q-rung orthopair FS
q-CDNNIVWA	q-CDNNIV weighted averaging
q-CDNNIVWG	q-CDNNIV weighted geometric
q-GCDNNIVWA	q-generalized CDNNIVWA
q-GCDNNIVWG	q-generalized CDNNIVWG.

safer human–robot interactions. Meanwhile, AI-powered robots are increasingly being employed for more complex tasks, such as autonomous decision-making, material handling, and visual inspection. As automation technologies continue to evolve, industrial robotics is playing a critical role in the development of smart factories and digital manufacturing ecosystems (Dimitropoulos et al., 2024). These robots are capable of processing large volumes of data, adapting to fluctuating conditions, and assisting decision-makers in selecting optimal operational strategies amid uncertainty.

The use of robots in industrial settings is driven by significant advances in engineering science and information technology, enabling them to perform precise, repetitive, and hazardous tasks with enhanced efficiency. Many industries, including those focused on welding, energy efficiency, and dynamic control, are increasingly adopting robots for a variety of applications. Rakkiyappan et al. (2014) explored the development of fuzzy neutral systems with recurring uncertainties, while Aslam et al. (2024) recently presented delayed networked fuzzy systems to address complex decision-making scenarios. As systems in the real world become more intricate, it becomes progressively harder for decision-makers to choose the best course of action. Nonetheless, with the help of MADM approaches, even in situations where multiple conflicting objectives must be balanced, businesses can still identify the best options to move forward.

1.1. Literature review and applications

To address data uncertainties, Zadeh originally introduced the concept of a fuzzy set (FS), in which each element is characterized by its truth degree (TD). An essential tool for managing fuzzy data that has rapidly expanded and is widely used across many domains. Over time, researchers have developed a variety of modifications of the original FSs, including interval-valued FSs and fuzzy soft sets. FS can be applied in DM to model MADM problems with imprecise information. Chen (2000) extended an MADM method using TOPSIS to solve fuzzy environments. However, FSs are limited to focusing on vague parameters or events based on the TD. Imprecise parameters are not handled correctly due to false degree (FD) and indeterminacy degree (ID). For instance, Atanassov introduced an IFS by defining an FD and a TD such that their sum is less than or equal to 1. Yager extends this sum condition to the square of its sum is not more than one, and hence introduces the set named Pythagorean fuzzy set (PFS) (Yager, 2013). Gundogdu and Kahraman (2019) introduced the idea of spherical FS (SFS) and the theory that goes along with it. This model is one of the new extensions of the FS theory, defined by a triple membership structure comprising an MG, an NMG, and a hesitancy function whose sum of squares is equal to or less than one. Compared to PFSs, the SFS model is more effective at handling ambiguity, imprecision, and uncertainty. Studies on SFSs are on the rise, according to a recent analysis of the most recent literature.

Various researchers have applied these extensions to develop techniques for solving different problems, such as clustering fuzzy c-numbers (Yang and Ko, 1996) and performing regression predictions using fuzzy time series (Xu and Li, 2001). Peng and Yang (2015) introduced interval-valued PFS (IVPFS), while Rahman et al. (2017) utilized geometric AOs for group DM based on the characteristics of IVPFS. Tariq et al. (2020) deals with the combination of AHP and TOPSIS methods for the ranking of security controls. Khan (2019) created an application that supports MAGDM employing Einstein's Choquet integral operators. Rahman et al. (2018) explored the induced feature of IVPFS, and Liu et al. (2020b) investigated a q-picture FS based on AO. Yang and Chang (2020) contributed to the development of IVPFS. Ramot et al. (2002) introduced the concept of complex FSs (CFSs), which extend FSs by moving from real-numbered TDs to complex numbers. The real numbers from a subset of the complex numbers, so CFS extends the range of FSs from $[0, 1]$ to a unit circle in the complex plane. For each $u \in U$, a complex value within this unit circle in the complex plane is assigned, and the truth function of CFS X is defined as $\mu_X(u)$. Thus, $\mu_X(u) = r_X(u)e^{i\wp X(u)}$, where $i = \sqrt{1}$, and all $\mu_X(u)$ values lie within the unit circle. It is stated that $r_X(u)$ is an amplitude term and $\wp X(u)$ is a phase term. $r_X(u) \in [0, 1]$ is the actual value of both of these terms. The representation of the CFS X is $\{(u, \mu_X(u)) | u \in U\}$.

Ramot et al. (2003) extended the concept of FSs to CFSs. Yazdanbakhsh and Dick (2015) further developed CFSs by analyzing their logical framework. Alkouri and Salleh (2012) incorporated FD into complex IFS (CIFS), allowing the TD and FD to be generalized across the entire unit circle. Ullah et al. (2020) introduced CPFS, and Liu et al. (2020a) first proposed complex root FSs. Rong et al. (2020) developed the complex orthopair FS (CROFS) using Maclaurin symmetric means. Yager (2016) introduced q-orthopair FSs (q-ROFSs), where the sum of the q-powers of the TD and FD is restricted to the interval $[0,1]$, as demonstrated by examples such as $0.85^4 + 0.754 = 0.52 + 0.32 = 0.84$. Compared to IFSs or PFSs, q-ROFS provides a more versatile model, making it especially useful for handling problematic fuzzy data across applications such as AOs, similarity measures, and hybrid AOs. The development of AOs within the q-ROFS framework has been notably successful. For instance, Fei and Deng (2020) created AOs based on MCDM based on group decision. Senapati et al. (2023) introduced Pythagorean fuzzy Aczel–Alsina average AOs. Zhou et al. (2020) proposed a divergence measure for PFSs based on faith functions, which has been applied in medical diagnostics. Similarity measures are essential for comparing items and evaluating their degrees in various scenarios. Khan et al. (2025) utilized Pythagorean hesitant fuzzy Sugeno–Weber AOs to assess uncertainties in the stock market. Song et al. (2020) applied the Pythagorean fuzzy analytic hierarchy process for loan risk assessment. Liu and Wang (2020) explored several q-orthopair fuzzy AOs and their applications to MADM. Real-world situations often involve complicated conditions, and data measurement techniques, such as similarity, separation, and entropy, help manage ambiguous data for tasks like clustering, classification, and pattern recognition. Despite the advantages of PFSs and q-ROFSs in MADM approaches, these methods exhibit certain limitations when handling fuzzy data. Seikh and Dey (2025) and Mandal and Seikh (2025) introduced q-orthopair FS and their extension with AOs, while Seikh and Mukherjee (2024) proposed the Fermatean fuzzy CRADIS technique. Our study introduces an MADM approach based on q-CDNNIVS, weighted AOs, score, and accuracy functions, offering valuable contributions to decision-makers in managing uncertain data.

In numerous applications, such as information fusion, data from multiple sensors is integrated. DM challenges arise in environments such as an interval spherical fuzzy setting (Bouraima et al., 2025) and the CRITIC-COPRAS method (Basuri et al., 2025). When the TD, FD, and reference parameters for objects in a universal set are represented by CFS, this leads to the generalization form FSs, IFSs, PFSs, and q-ROFS. Complex values frequently occur in real-world scenarios and must be properly managed to achieve the desired results. The concept of CLDFS seeks to overcome these challenges. This is one of the key contributions of CLDFS. Akram et al. (2020) proposed a new theory of complex picture FSs, based on Hamacher AOs. The CPFS model incorporates degrees of truth, abstention, and falsity, each represented by an amplitude and a phase term. The hybridizations of Archimedean and Aczel–Alsina AOs are also discussed in Naseem et al. (2023) and Khan et al. (2023). Neutrosophic sets (NSs) are characterized by components in the range $[0,1]$ based on TD, ID and FD (Smarandache, 1999), and this neutrality FSs apart from both IFSs and neutrosophic systems. Palanikumar et al. (2022, 2023) introduced the Pythagorean neutrosophic normal interval-valued set and extended SFS as a novel framework. NS is a generalization of crisp, FS, IVFS, IFS, IVIFS, etc. Engineering and scientific applications of NS are challenging to implement directly. Zhang et al. presented an investigation of MCDM under interval NS with information (Zhang et al., 2014). Jana and Pal (2019) presented AOs in single-valued neutrosophic soft sets. A broad range of MADM problems can be modeled using an IVNS-based TOPSIS method (Chi and Liu, 2013).

In the CIFS and CPFS structures, the complex-valued TD and FD display unique interdependent characteristics. Complex-valued reference parameters are integral to the CLDFS framework, setting it apart from both CIFS and CPFS. By altering the interpretation of these complex-valued reference parameters, an alternative classification of the objects can be achieved. Rahman and Muhammad (2024) introduced a complex polytopic fuzzy model. Riaz and Farid (2023) examined linear Diophantine fuzzy soft-max AOs. Liu et al. (2026) developed a linear Diophantine fuzzy hybrid decision support system for evaluating sustainability in renewable energy resources, which shares key similarities with the current approach in addressing decision-making challenges. Al-Quran et al. (2023, 2024) contributed significantly to the advancement of q-complex neutrosophic sets and their applications in decision-making, emphasizing the role of axiomatic properties and AOs, directly relevant to the methodologies presented in this paper. Furthermore, Al-Qudah and Al-Sharqi (2023) proposed a decision-making algorithm based on similarity measures in possibility interval-valued neutrosophic soft settings, thereby enhancing the practical use of neutrosophic sets in decision-making. Additionally, M et al. (2024) evaluated AI-driven solid waste segregation technologies using complex q-rung picture fuzzy Frank AOs, while Petchimuthu et al. (2025) applied complex q-rung picture fuzzy SW operators to support strategies for reducing greenhouse gas emissions, highlighting the value of complex fuzzy frameworks in environmental and engineering issues (M et al., 2024; Petchimuthu et al., 2025). These studies emphasize the increasing complexity of decision environments and the growing need for advanced methods such as q-CDNNIVS to better manage uncertainty, ambiguity, and interdependent data.

1.2. Explanation of the truth degrees in decision-making

In this study, we introduce three independent truth degrees, TD, ID, and FD, within the framework of the q-CDNNIVS. These degrees are designed to handle and evaluate the uncertainty and ambiguity inherent in DM problems, particularly in real-world scenarios where data may be ambiguous or incomplete.

1. TD: Represents the confidence that a given decision or option satisfies the criterion. A higher TD indicates higher certainty in the truth of the decision.
2. ID: Reflects the uncertainty or lack of clarity in the decision-making method. A high ID indicates greater ambiguity or insufficient information.
3. FD: Indicates the extent to which the decision or option fails to meet the required criterion. A higher FD means the option is farther from the desired outcome.

These three degrees collectively offer a holistic assessment of the available alternatives in decision-making. To better understand their roles and how decision-makers can interpret these degrees in practical contexts, we present Table 1.

1.3. Motivation of the study

Vagueness, inconsistency, indeterminacy, and inadequate knowledge are common characteristics of real-world decision-making challenges. Experts are unable to explain their opinions in many real-world scenarios, particularly in MADM, using only exact numerical figures. Although classical fuzzy models and some of their extensions have provided useful tools for managing uncertainty, they nevertheless have drawbacks when the data is highly ambiguous, interval-valued, or involves more complex structures. Specifically, the coexistence of truth, indeterminacy, and falsity, as well as the flexibility demanded by contemporary decision contexts, may not be sufficiently captured by current models.

This project was driven by the need to develop a more expressive and flexible framework to capture uncertainty in complex MADM scenarios. Many existing approaches, including FSs, IFSs, PFSs, and q-ROFSs, are unable to fully address decision situations involving interval-valued assessments, normal fuzzy information, and independent neutrosophic components in a unified manner. This is the motivation behind the proposed q-CDNNIVS. Additionally, there is an increasing need for aggregation methods that produce trustworthy ranks and retain more information as decision problems become more complex. Thus, the following factors serve as motivation for this study:

Table 1
Interpretation of the truth degrees in decision-making.

Degree type	Role in decision-making	Practical interpretation
TD	Represents the confidence that a given decision satisfies the criterion	High TD indicates confidence that the decision meets the criterion
ID	Reflects uncertainty or lack of uncertainty in the available data	High ID indicates ambiguity or insufficient information for clear decision-making
FD	Indicates the degree to which an option fails to meet the criteria	High FD signifies that the option does not fulfill the criterion well

(i) Existing fuzzy and neutrosophic models have limited ability to represent highly complex uncertainty, particularly when truth, indeterminacy, and falsity information must be handled simultaneously in interval-valued form.

(ii) The flexibility of decision modeling needs to be improved so that decision-makers can express their evaluations more realistically under varying levels of ambiguity and inconsistency.

(iii) There is a need for suitable AOs that can combine q-CDNNIV information in a mathematically consistent and meaningful way for MADM applications.

(iv) A reliable MADM framework is required for practical selection problems, where multiple conflicting attributes must be evaluated together under uncertain conditions.

(v) Industrial robot selection provides an appropriate real-world motivation for the proposed model, since it involves several competing criteria such as reach, payload, flexibility, speed, and standardization, all of which require robust evaluation under uncertainty.

For these reasons, this study develops a new q-CDNNIVS-based MADM framework and investigates its usefulness through AOs, ranking analysis, and an application to industrial robot selection.

1.4. Contribution of the study

This study makes significant strides in advancing decision-making, especially in managing uncertainty and ambiguity in complex issues. The following significant contributions are made:

(1) This study introduces the concept of q-CDNNIVS, which provides a novel framework for managing uncertainty and ambiguity in decision-making processes involving complex, uncertain, and inconsistent data.
(2) Several AOs, including weighted averaging (WA) and weighted geometric (WG) operators, are proposed; their fundamental characteristics, such as commutativity, idempotency, and associativity, are thoroughly explored. These operators enhance the robustness of decision-making in uncertain environments.
(3) A MADM method is developed to address complex decision-making issues. The method facilitates the evaluation of alternatives based on multiple criteria, enhancing the ability to choose an optimal solution in multifaceted scenarios.
(4) The proposed method is applied in a real-world case study focused on industrial robot selection, highlighting the algorithm's practical applicability and effectiveness in addressing real-world decision-making challenges.
(5) A comparative evaluation with existing decision-making methods is also conducted to assess the performance of the proposed method. The findings reveal that the proposed method surpasses others in terms of precision, adaptability, and reliability, especially when handling complex and inconsistent data.

Additionally, Fig. 1 highlights the uniqueness of the proposed q-CDNNIVS by displaying a map of current fuzzy extensions in addition to the new extension.

1.5. Structure of the study

This study is structured into eleven sections. Section 2 introduces the preliminary findings. Section 3 details the operations on the q-CDNNIVS, focusing on the mathematical processes for managing complex data. Section 4 introduces various AOs (q-CDNNIVWA, q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG) and discusses their properties, such as commutativity and idempotency. Section 5 outlines the MADM method based on q-CDNNIVS, explaining how the algorithm integrates these operators to solve decision-making problems. Section 6 presents a case study on industrial robot selection, detailing the problem description and the definition of alternatives in Section 6.1. Section 6.2 presents the solution to the problem, applying the proposed methodology to the case study. Section 7 offers a detailed comparison analysis, contrasting the proposed method with other established decision-making techniques to highlight its unique strengths. Section 8 provides a comprehensive sensitivity analysis. Section 8.1 examines the effect of the parameter q, while Section 8.2 analyzes the influence of variations in criteria weights on the ranking stability. Section 9 outlines the key advantages of the study, providing a summary of the methodology's benefits and its potential applications across different fields. Finally, Section 10 wraps up the study by summarizing the major contributions and proposing avenues for future research, particularly in healthcare, business, and manufacturing, where complex fuzzy environments may pose new challenges and opportunities for applying this approach.

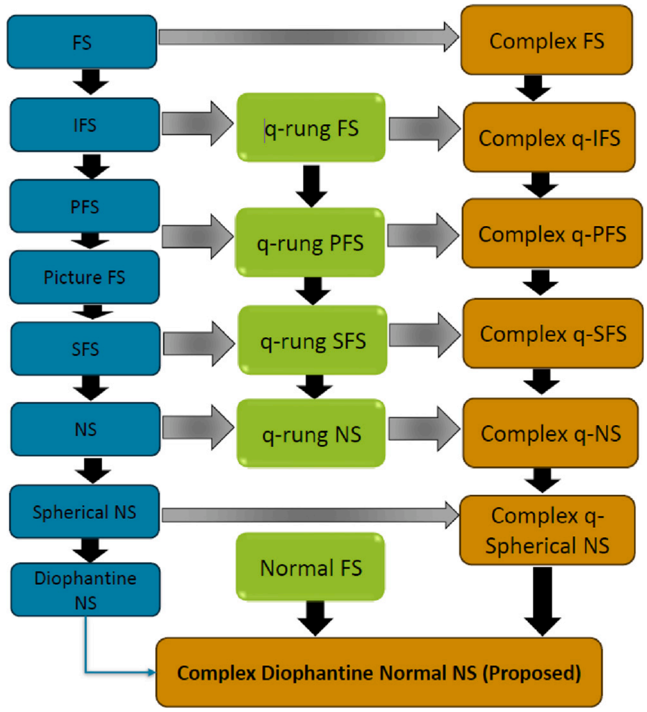

Fig. 1. Approaches to fuzzy extensions.

2. Background

In this section, we will explore fundamental concepts relevant to the study and provide the theoretical underpinnings for the methodology presented. A solid understanding of these concepts is essential for comprehending the subsequent developments and their applications in decision-making under uncertainty. We begin by introducing key definitions and frameworks that underpin our approach. Let U denote the universal set.

Definition 2.1 (Yager, 2013). The PFS $\Phi = \left\{a, \left\langle \mu^T(a), \mu^F(a)\right\rangle \middle| a \in U\right\}$, since $\mu^T, \mu^F : U \to [0,1]$ denote the TD and FD of $a \in U$ to Φ, respectively and $0 \le (\mu^T(a))^2 + (\mu^F(a))^2 \le 1$. For $\Phi = \left\langle \mu^T, \mu^F \right\rangle$ is called a Pythagorean fuzzy number.

Definition 2.2 (Ni et al., 2025). The q-FS $\Phi = \left\{a, \left\langle \mu^T(a), \mu^F(a)\right\rangle \middle| a \in U\right\}$, since $\mu^T, \mu^F : U \to [0,1]$ denote the TD and FD of $a \in U$ to Φ, respectively and $0 \le (\mu^T(a))^q + (\mu^F(a))^q \le 1$. For $\Phi = \left\langle \mu^T, \mu^F \right\rangle$ is called a q-fuzzy number.

Definition 2.3 (Peng and Yang, 2015). The IVPFS $\widetilde{\Phi} = \left\{a, \left\langle \widetilde{\mu^T}(a), \widetilde{\mu^F}(a)\right\rangle \middle| a \in U\right\}$, where $\widetilde{\mu^T}, \widetilde{\mu^F} : U \to Int([0,1])$ denote the TD and FD of $a \in U$ to Φ, respectively, and $0 \le (\alpha\mu^{Tu}(a))^2 + (\mu^{Fu}(a))^2 \le 1$. For $\Phi = \left\langle \left[\mu^{Tl}, \mu^{Tu}\right], \left[\mu^{Fl}, \mu^{Fu}\right] \right\rangle$ is called a Pythagorean interval-valued fuzzy number (PIVFN).

Definition 2.4 (Gundogdu and Kahraman, 2019). The SFS $\Phi = \left\{a, \left\langle \mu^T(a), \mu^I(a), \mu^F(a)\right\rangle \middle| u \in U\right\}$, since $\mu^T : U \to [0,1]$, $\mu^I : U \to [0,1]$ and $\mu^F : U \to [0,1]$ is denote the TD, ID and FD of $u \in U$ to $\widetilde{\Phi}$, respectively and $0 \le (\mu^T(a))^2 + (\mu^I(a))^2 + (\mu^F(a))^2 \le 1$. For convenience, $\widetilde{\Phi} = \left\langle \mu^T, \mu^I, \mu^F \right\rangle$ is called a spherical fuzzy number (SFN).

Definition 2.5 (Smarandache, 1999). The NS $\Phi = \left\{a, \left\langle \mu^T(a), \mu^I(a), \mu^F(a)\right\rangle \middle| a \in U\right\}$, since $\mu^T, \mu^I, \mu^F : U \to [0,1]$ denote the TD, ID and FD of $a \in U$ to Φ, respectively and $0 \le \mu^T(a) + \mu^I(a) + \mu^F(a) \le 3$. For $\Phi = \left\langle \mu^T, \mu^I, \mu^F \right\rangle$ is called a neutrosophic number (NN).

Definition 2.6 (Palanikumar et al., 2022). The Pythagorean NS $\Phi = \left\{a, \left\langle \mu^T(a), \mu^I(a), \mu^F(a)\right\rangle \middle| a \in U\right\}$, since $\mu^T, \mu^I, \mu^F : U \to [0,1]$ denote the TD, ID and FD of $a \in U$ to Φ, respectively and $0 \le (\mu^T(a))^2 + (\mu^I(a))^2 + (\mu^F(a))^2 \le 2$. For $\Phi = \left\langle \mu^T, \mu^I, \mu^F \right\rangle$ is called a Pythagorean NN.

Definition 2.7 (Peng and Yang, 2015). Let $\Phi = \left\langle [\mu^{Tl}, \mu^{Tu}], [\mu^{Fl}, \mu^{Fu}] \right\rangle$, $\Phi_1 = \left\langle [\mu_1^{Tl}, \mu_1^{Tu}], [\mu_1^{Fl}, \mu_1^{Fu}] \right\rangle$ and $\Phi_2 = \left\langle [\mu_2^{Tl}, \mu_2^{Tu}], [\mu_2^{Fl}, \mu_2^{Fu}] \right\rangle$ be any three PIVFNs, and $q > 0$. Then

1. $$\Phi_1 \bigoplus \Phi_2 = \begin{bmatrix} \left[\sqrt{(\mu_1^{Tl})^2 + (\mu_2^{Tl})^2 - (\mu_1^{Tl})^2 \cdot (\mu_2^{Tl})^2}, \sqrt{(\mu_1^{Tu})^2 + (\mu_2^{Tu})^2 - (\mu_1^{Tu})^2 \cdot (\mu_2^{Tu})^2}\right], \\ \left[\mu_1^{Fl} \cdot \mu_2^{Fl}, \mu_1^{Fu} \cdot \mu_2^{Fu}\right] \end{bmatrix},$$
2. $$\Phi_1 \bigotimes \Phi_2 = \begin{bmatrix} \left[\mu_1^{Tl} \cdot \mu_2^{Tl}, \mu_1^{Tu} \cdot \mu_2^{Tu}\right], \\ \left[\sqrt{(\mu_1^{Fl})^2 + (\mu_2^{Fl})^2 - (\mu_1^{Fl})^2 \cdot (\mu_2^{Fl})^2}, \sqrt{(\mu_1^{Fu})^2 + (\mu_2^{Fu})^2 - (\mu_1^{Fu})^2 \cdot (\mu_2^{Fu})^2}\right] \end{bmatrix},$$

3. $q \cdot \Phi = \left[\left[\sqrt{1-\left(1-(\mu^{Tl})^2\right)^q}, \sqrt{1-\left(1-(\mu^{Tu})^2\right)^q}\right], \left[(\mu^{Fl})^q, (\mu^{Fu})^q\right]\right]$,
4. $\Phi^q = \left[\left[(\mu^{Tl})^q, (\mu^{Tu})^q\right], \left[\sqrt{1-\left(1-(\mu^{Fl})^2\right)^q}, \sqrt{1-\left(1-(\mu^{Fu})^2\right)^q}\right]\right]$.

Definition 2.8 (Yang and Chang, 2020)**.** Let $\Phi = \left\langle [\mu^{Tl}, \mu^{Tu}], [\mu^{Fl}, \mu^{Fu}] \right\rangle$ be the PIVFN. Then the score function of Φ is calculated as

$$S(\Phi) = \frac{1}{2}\left((\mu^{Tl})^2 + (\mu^{Tu})^2 - (\mu^{Fl})^2 - (\mu^{Fu})^2\right), \ S(\Phi) \in [-1, 1]$$

The accuracy function of Φ is calculated as

$$H(\Phi) = \frac{1}{2}\left((\mu^{Tl})^2 + (\mu^{Tu})^2 + (\mu^{Fl})^2 + (\mu^{Fu})^2\right), \ H(\Phi) \in [0, 1].$$

Definition 2.9 (Yang and Ko, 1996 *and* Xu and Li, 2001)**.** (i) Let $M(a) = e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, (\tau > 0)$ be the fuzzy number (FN), whenever $M = (\vartheta, \tau)$ is mentioned as a normal FN (NFN), since N is denoted as a normal FS.

(ii) Let $L = (\vartheta_1, \tau_1) \in N$ and $M = (\vartheta_2, \tau_2) \in N$, $(\tau_1, \tau_2 > 0)$. Then $\mathbb{D}(L, M) = \sqrt{(\vartheta_1 - \vartheta_2)^2 + \frac{1}{2}(\tau_1 - \tau_2)^2}$ is referred as a distance between L and M.

3. Construction of q-CDNNIVNs

This section introduces some new operations and defines aggregate operators for q-CDNNIVN. Robotics plays a significant role in several decision-based applications. Our exploration of linguistic developments, fuzzy numbers, and distance measures is further enhanced by these fundamental characteristics. Through q-CDNIVNs and NFNs, q-CDNNIVNs and their operations were defined. In addition to the score and accuracy values of the q-CDNNIVNs, a few mathematical properties were introduced.

Definition 3.1. Let $(\vartheta, \tau) \in N$, $\widetilde{\Phi} = \left\langle (\vartheta, \tau); [R^{Tl}e^{i2\pi J^{Tl}}, R^{Tu}e^{i2\pi J^{Tu}}], [R^{Il}e^{i2\pi J^{Il}}, R^{Iu}e^{i2\pi J^{Iu}}], [R^{Fl}e^{i2\pi J^{Fl}}, R^{Fu}e^{i2\pi J^{Fu}}], [\alpha, \beta, \gamma] \right\rangle$ is a q-CDNNIVN. The TD, ID and FD are defined as
$\left[R^{Tl}, R^{Tu}\right] = \left[R^{Tl}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, R^{Tu}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$, $\left[R^{Il}, R^{Iu}\right] = \left[R^{Il}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, R^{Iu}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$ and $\left[R^{Fl}, R^{Fu}\right] = \left[1-\left(1-R^{Fl}\right)e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, 1-\left(1-R^{Fu}\right)e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$, respectively. Also, $\left[J^{Tl}, J^{Tu}\right] = \left[J^{Tl}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, J^{Tu}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$, $\left[J^{Il}, J^{Iu}\right] = \left[J^{Il}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, J^{Iu}e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$ and $\left[J^{Fl}, J^{Fu}\right] = \left[1-\left(1-J^{Fl}\right)e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}, 1-\left(1-J^{Fu}\right)e^{-\left(\frac{u-\vartheta}{\tau}\right)^2}\right]$, $u \in U$ respectively, where R^T denotes the real-valued function from X to the closed unit interval and e^{iJ^T} is a periodic function whose periodic law and principal period are 2π and $0 < \arg z(u) \le 2\pi$, respectively. Note that $J^T(u) = \arg z(u) + 2k\pi$, k is a integer and $\arg z(u)$ is the principal argument, where U is a non-empty set and $\left[R^{Tl}, R^{Tu}\right], \left[R^{Il}, R^{Iu}\right], \left[R^{Fl}, R^{Fu}\right], \left[J^{Tl}, J^{Tu}\right], \left[J^{Il}, J^{Iu}\right], \left[J^{Fl}, J^{Fu}\right] \in Int([0, 1])$ and $0 \le \left(\alpha R^{Tu}(u)\right)^q + \left(\beta R^{Iu}(u)\right)^q + \left(\gamma R^{Fu}(u)\right)^q \le 1$ and $0 \le \left(\alpha J^{Tu}(u)\right)^q + \left(\beta J^{Iu}(u)\right)^q + \left(\gamma J^{Fu}(u)\right)^q \le 1$.

Definition 3.2. The q-CDNNIVN $\widetilde{\Phi} = \left\langle (\vartheta, \tau); [R^{Tl}e^{i2\pi J^{Tl}}, R^{Tu}e^{i2\pi J^{Tu}}], [R^{Il}e^{i2\pi J^{Il}}, R^{Iu}e^{i2\pi J^{Iu}}], [R^{Fl}e^{i2\pi J^{Fl}}, R^{Fu}e^{i2\pi J^{Fu}}], [\alpha, \beta, \gamma] \right\rangle$, the score function is defined as $S_1(\widetilde{\Phi}) = \frac{\vartheta}{2}(C + D)$, $S_2(\widetilde{\Phi}) = \frac{\tau}{2}(C + D)$ and $S(\widetilde{\Phi}) = \frac{S_1(\widetilde{\Phi}) + S_2(\widetilde{\Phi})}{2}$. where

$$C = \left(\frac{(\alpha R^{Tl})^2 + (\alpha R^{Tu})^2}{2} - \frac{(\beta R^{Il})^2 + (\beta R^{Iu})^2}{2} + 2 - \frac{(\gamma R^{Fl})^2 + (\gamma R^{Fu})^2}{2}\right)$$

$$D = \left(\frac{(\alpha J^{Tl})^2 + (\alpha J^{Tu})^2}{2} - \frac{(\beta J^{Il})^2 + (\beta J^{Iu})^2}{2} - \frac{(\gamma J^{Fl})^2 + (\gamma J^{Fu})^2}{2}\right),$$

where $S(\widetilde{\Phi}) \in [-1, 1]$.

The accuracy function $H_1(\widetilde{\Phi}) = \frac{\vartheta}{2}\left(C_1 + D_1\right)$, $H_2(\widetilde{\Phi}) = \frac{\tau}{2}\left(C_1 + D_1\right)$ and $H(\widetilde{\Phi}) = \frac{H_1(\widetilde{\Phi}) + H_2(\widetilde{\Phi})}{2}$, where

$$C_1 = \left(\frac{(\alpha R^{Tl})^2 + (\alpha R^{Tu})^2}{2} + \frac{(\beta R^{Il})^2 + (\beta R^{Iu})^2}{2} + \frac{(\gamma R^{Fl})^2 + (\gamma R^{Fu})^2}{2}\right)$$

$$D_1 = \left(\frac{(\alpha J^{Tl})^2 + (\alpha J^{Tu})^2}{2} + \frac{(\beta J^{Il})^2 + (\beta J^{Iu})^2}{2} + \frac{(\gamma J^{Fl})^2 + (\gamma J^{Fu})^2}{2}\right),$$

where $H(\widetilde{\Phi}) \in [0, 1]$, where C and C_1 are called the value of amplitude TD, ID, and FD. Also, D and D_1 are called the value of phase TD, ID, and FD of Φ.

Fig. 2 illustrates the comparison rule based on the score and accuracy functions of two q-CDNNIVNs. The score values are first used to determine the preferred alternative; when two score values are equal, the accuracy value provides an additional criterion for ranking. If both score and accuracy values are identical, the two q-CDNNIVNs are considered indistinguishable.

Remark 3.1. Assume that $q = 1$. According to Definition 3.2,

a. CDNNIVS is changed to CNNIVS if $(\alpha, \beta, \gamma) = (1, 1, 1)$.
b. CNNIVS is changed to NNIVS if the amplitude terms are eliminated.
c. The NNIVS is transformed into the PIVFNS if the degree of indeterminacy is eliminated.

Definition 3.3. Let $\widetilde{\Phi} = \left\langle (\vartheta, \tau); [R^{Tl}e^{i2\pi J^{Tl}}, R^{Tu}e^{i2\pi J^{Tu}}], [R^{Il}e^{i2\pi J^{Il}}, R^{Iu}e^{i2\pi J^{Iu}}], [R^{Fl}e^{i2\pi J^{Fl}}, R^{Fu}e^{i2\pi J^{Fu}}], [\alpha, \beta, \gamma] \right\rangle$, $\widetilde{\Phi}_1 = \left\langle (\vartheta_1, \tau_1); [R_1^{Tl}e^{i2\pi J_1^{Tl}}, R_1^{Tu}e^{i2\pi J_1^{Tu}}], [R_1^{Il}e^{i2\pi J_1^{Il}}, R_1^{Iu}e^{i2\pi J_1^{Iu}}], [R_1^{Fl}e^{i2\pi J_1^{Fl}}, R_1^{Fu}e^{i2\pi J_1^{Fu}}], [\alpha_1, \beta_1, \gamma_1] \right\rangle$ and $\widetilde{\Phi}_2 = \left\langle (\vartheta_2, \tau_2); [R_2^{Tl}e^{i2\pi J_2^{Tl}}, R_2^{Tu}e^{i2\pi J_2^{Tu}}], [R_2^{Il}e^{i2\pi J_2^{Il}}, R_2^{Iu}e^{i2\pi J_2^{Iu}}], [R_2^{Fl}e^{i2\pi J_2^{Fl}}, R_2^{Fu}e^{i2\pi J_2^{Fu}}], [\alpha_2, \beta_2, \gamma_2] \right\rangle$ be the any three q-CDNNIVNs, and $q > 0$. Then

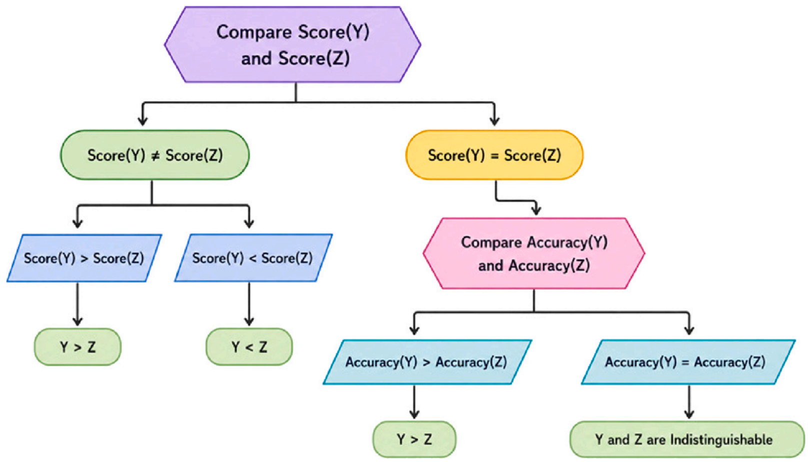

Fig. 2. Scope of the score and accuracy values.

1. $$\widetilde{\Phi}_1 \bigoplus \widetilde{\Phi}_2 = \begin{bmatrix} (\vartheta_1+\vartheta_2, \tau_1+\tau_2); \\ \begin{bmatrix} \sqrt[2q]{(\alpha R_1^{Tl})^{2q} + (\alpha R_2^{Tl})^{2q} - (\alpha R_1^{Tl})^{2q}\cdot(\alpha R_2^{Tl})^{2q}}\, e^{i2\pi \sqrt[2q]{(\alpha J_1^{Tl})^{2q}+(\alpha J_2^{Tl})^{2q}-(\alpha J_1^{Tl})^{2q}\cdot(\alpha J_2^{Tl})^{2q}}}, \\ \sqrt[2q]{(\alpha R_1^{Tu})^{2q} + (\alpha R_2^{Tu})^{2q} - (\alpha R_1^{Tu})^{2q}\cdot(\alpha R_2^{Tu})^{2q}}\, e^{i2\pi \sqrt[2q]{(\alpha J_1^{Tu})^{2q}+(\alpha J_2^{Tu})^{2q}-(\alpha J_1^{Tu})^{2q}\cdot(\alpha J_2^{Tu})^{2q}}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[q]{(\beta R_1^{Il})^{q} + (\beta R_2^{Il})^{q} - (\beta R_1^{Il})^{q}\cdot(\beta R_2^{Il})^{q}}\, e^{i2\pi \sqrt[q]{(\beta J_1^{Il})^{q}+(\beta J_2^{Il})^{q}-(\beta J_1^{Il})^{q}\cdot(\beta J_2^{Il})^{q}}}, \\ \sqrt[q]{(\beta R_1^{Iu})^{q} + (\beta R_2^{Iu})^{q} - (\beta R_1^{Iu})^{q}\cdot(\beta R_2^{Iu})^{q}}\, e^{i2\pi \sqrt[q]{(\beta J_1^{Iu})^{q}+(\beta J_2^{Iu})^{q}-(\beta J_1^{Iu})^{q}\cdot(\beta J_2^{Iu})^{q}}} \end{bmatrix}, \\ \left[\left(\gamma R_1^{Fl}\cdot \gamma R_2^{Fl}\right) e^{i2\pi(\gamma J_1^{Fl}\cdot\gamma J_2^{Fl})}, \left(\gamma R_1^{Fu}\cdot\gamma R_2^{Fu}\right) e^{i2\pi(\gamma J_1^{Fl}\cdot \gamma J_2^{Fl})}\right] \end{bmatrix},$$

2. $$\widetilde{\Phi}_1 \bigotimes \widetilde{\Phi}_2 = \begin{bmatrix} (\vartheta_1\cdot\vartheta_2, \tau_1\cdot\tau_2); \\ \left[\left(\alpha R_1^{Tl}\cdot \alpha R_2^{Tl}\right) e^{i2\pi(\alpha J_1^{Tl}\cdot\alpha J_2^{Tl})}, \left(\alpha R_1^{Tu}\cdot\alpha R_2^{Tu}\right) e^{i2\pi(\alpha J_1^{Tl}\cdot \alpha J_2^{Tl})}\right], \\ \begin{bmatrix} \sqrt[q]{(\beta R_1^{Il})^{q} + (\beta R_2^{Il})^{q} - (\beta R_1^{Il})^{q}\cdot(\beta R_2^{Il})^{q}}\, e^{i2\pi \sqrt[q]{(\beta J_1^{Il})^{q}+(\beta J_2^{Il})^{q}-(\beta J_1^{Il})^{q}\cdot(\beta J_2^{Il})^{q}}}, \\ \sqrt[q]{(\beta R_1^{Iu})^{q} + (\beta R_2^{Iu})^{q} - (\beta R_1^{Iu})^{q}\cdot(\beta R_2^{Iu})^{q}}\, e^{i2\pi \sqrt[q]{(\beta J_1^{Iu})^{q}+(\beta J_2^{Iu})^{q}-(\beta J_1^{Iu})^{q}\cdot(\beta J_2^{Iu})^{q}}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[2q]{(\gamma R_1^{Fl})^{2q} + (\gamma R_2^{Fl})^{2q} - (\gamma R_1^{Fl})^{2q}\cdot(\gamma R_2^{Fl})^{2q}}, e^{i2\pi \sqrt[2q]{(\gamma J_1^{Fl})^{2q}+(\gamma J_2^{Fl})^{2q}-(\gamma J_1^{Fl})^{2q}\cdot(\gamma J_2^{Fl})^{2q}}}, \\ \sqrt[2q]{(\gamma R_1^{Fu})^{2q} + (\gamma R_2^{Fu})^{2q} - (\gamma R_1^{Fu})^{2q}\cdot(\gamma R_2^{Fu})^{2q}}\, e^{i2\pi \sqrt[2q]{(\gamma J_1^{Fu})^{2q}+(\gamma J_2^{Fu})^{2q}-(\gamma J_1^{Fu})^{2q}\cdot(\gamma J_2^{Fu})^{2q}}} \end{bmatrix} \end{bmatrix},$$

3. $$q\cdot\widetilde{\Phi} = \begin{bmatrix} (q\cdot\vartheta, q\cdot\tau); \\ \begin{bmatrix} \sqrt[2q]{1-\left(1-(\alpha R^{Tl})^{2q}\right)^q}\, e^{i2\pi \sqrt[2q]{1-\left(1-(\alpha J^{Tl})^{2q}\right)^q}}, \\ \sqrt[2q]{1-\left(1-(\alpha R^{Tu})^{2q}\right)^q}\, e^{i2\pi \sqrt[2q]{1-\left(1-(\alpha J^{Tu})^{2q}\right)^q}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[q]{1-\left(1-(\beta R^{Il})^{q}\right)^q}\, e^{i2\pi \sqrt[q]{1-\left(1-(\beta J^{Il})^{q}\right)^q}}, \\ \sqrt[q]{1-\left(1-(\beta R^{Iu})^{q}\right)^q}\, e^{i2\pi \sqrt[q]{1-\left(1-(\beta J^{Iu})^{q}\right)^q}} \end{bmatrix} \\ \left[(\gamma R^{Fl})^q e^{i2\pi(\gamma J^{Fl})^q}, (\gamma R^{Fu})^q e^{i2\pi(\gamma J^{Fu})^q}\right] \end{bmatrix},$$

4. $$\widetilde{\Phi}^q = \begin{bmatrix} (\vartheta^q, \tau^q); \\ \left[(\alpha R^{Tl})^q e^{i2\pi(\alpha J^{Tl})^q}, (\alpha R^{Tu})^q e^{i2\pi(\alpha J^{Tu})^q}\right], \\ \begin{bmatrix} \sqrt[q]{1-\left(1-(\beta R^{Il})^{q}\right)^q}\, e^{i2\pi \sqrt[q]{1-\left(1-(\beta J^{Il})^{q}\right)^q}}, \\ \sqrt[q]{1-\left(1-(\beta R^{Iu})^{q}\right)^q}\, e^{i2\pi \sqrt[q]{1-\left(1-(\beta J^{Iu})^{q}\right)^q}} \end{bmatrix}, \\ \begin{bmatrix} \sqrt[2q]{1-\left(1-(\gamma R^{Fl})^{2q}\right)^q}\, e^{i2\pi \sqrt[2q]{1-\left(1-(\gamma J^{Fl})^{2q}\right)^q}}, \\ \sqrt[2q]{1-\left(1-(\gamma R^{Fu})^{2q}\right)^q}\, e^{i2\pi \sqrt[2q]{1-\left(1-(\gamma J^{Fu})^{2q}\right)^q}} \end{bmatrix} \end{bmatrix}.$$

4. Weighted averaging and geometric operators for q-CDNNIVNs

Discussing the benefits of these AOs with q-CDNNIVS is the aim of this section. The decision maker can get a more comprehensive interpretation of the outcome by combining these operators with WA and WG operators. This section provides a brief overview of some of the families of q-CDNNIVWA, q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG operators. Associativity, monotonicity, commutativity, idempotency, and boundedness are the primary characteristics of these AOs.

4.1. q-CDNNIVWA

The operating laws for the q-CDNNIVS are defined as follows.

Definition 4.1. Let

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], \\ {[R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}]}, [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be the collection of q-CDNNIVNs, $W = (\wp_1, \wp_2, \dots, \wp_m)$ be the weight of $\widetilde{\Phi}_k$, $\wp_k \geq 0$ and $\bigoplus_{k=1}^m \wp_k = 1$. Then q-CDNNIVWA $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \dots, \widetilde{\Phi}_m) = \bigoplus_{k=1}^m \wp_k \widetilde{\Phi}_k$.

Theorem 4.1. *Let*

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], \\ {[R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}]}, [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. Then q-CDNNIVWA$(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \dots, \widetilde{\Phi}_m) =$

$$\left[\begin{array}{c} \left(\bigoplus_{k=1}^m \wp_k \vartheta_k, \bigoplus_{k=1}^m \wp_k \tau_k\right); \\ \left[\sqrt[2q]{1 - \odot_{k=1}^m \left(1 - (\alpha R_k^{Tl})^{2q}\right)^{\wp_k}} e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^m \left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}}}, \right. \\ \left. \sqrt[2q]{1 - \odot_{k=1}^m \left(1 - (\alpha R_k^{Tu})^{2q}\right)^{\wp_k}} e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^m \left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \right]; \\ \left[\sqrt[q]{1 - \odot_{k=1}^m \left(1 - (\beta R_k^{Il})^{q}\right)^{\wp_k}} e^{i2\pi \sqrt[q]{1-\odot_{k=1}^m \left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}}}, \right. \\ \left. \sqrt[q]{1 - \odot_{k=1}^m \left(1 - (\beta R_k^{Iu})^{q}\right)^{\wp_k}} e^{i2\pi \sqrt[q]{1-\odot_{k=1}^m \left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}}} \right]; \\ \left[\odot_{k=1}^m (\gamma R_k^{Fl})^{\wp_k} e^{i2\pi \odot_{k=1}^m \left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}}, \odot_{k=1}^m (\gamma R_k^{Fu})^{\wp_k} e^{i2\pi \odot_{k=1}^m \left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}} \right] \end{array} \right].$$

(Associative property)

Proof. The proof of Theorem 4.1 is proposed in Appendix A. □

Theorem 4.2. *If all $\widetilde{\Phi}_k = \left\langle (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], [R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \dots, m)$ are equal with $\widetilde{\Phi}_k = \widetilde{\Phi}$, then q-CDNNIVWA$(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \dots, \widetilde{\Phi}_m) = \widetilde{\Phi}$. (Idempotency property)*

Proof. The proof of Theorem 4.2 is proposed in Appendix B. □

Theorem 4.3. *Let $\widetilde{\Phi}_k = \left\langle (\vartheta_{kj}, \tau_{kj}); [R_{kj}^{Tl} e^{i2\pi J_{kj}^{Tl}}, R_{kj}^{Tu} e^{i2\pi J_{kj}^{Tu}}], [R_{kj}^{Il} e^{i2\pi J_{kj}^{Il}}, R_{kj}^{Iu} e^{i2\pi J_{kj}^{Iu}}], [R_{kj}^{Fl} e^{i2\pi J_{kj}^{Fl}}, R_{kj}^{Fu} e^{i2\pi J_{kj}^{Fu}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \dots, m); (j = 1, 2, \dots, k_j)$ be a collection of q-CDNNIVWA, where*
$\underline{\vartheta} = \min \vartheta_{kj}$, $\overline{\vartheta} = \max \vartheta_{kj}$, $\underline{\tau} = \max \tau_{kj}$, $\overline{\tau} = \min \tau_{kj}$, $\underline{\alpha R^{Tl}} = \min \alpha R_{kj}^{T-}$, $\overline{\alpha R^{Tl}} = \max \alpha R_{kj}^{T-}$, $\underline{\alpha R^{Tu}} = \min \alpha R_{kj}^{T+}$, $\overline{\alpha R^{Tu}} = \max \alpha R_{kj}^{T+}$, $\underline{\beta R^{Il}} = \min \beta R_{kj}^{I-}$, $\overline{\beta R^{Il}} = \max \beta R_{kj}^{I-}$, $\underline{\beta R^{Iu}} = \min \beta R_{kj}^{I+}$, $\overline{\beta R^{Iu}} = \max \beta R_{kj}^{I+}$, $\underline{\gamma R^{Fl}} = \min \gamma R_{kj}^{F-}$, $\overline{\gamma R^{Fl}} = \max \gamma R_{kj}^{F-}$, $\underline{\gamma R^{Fu}} = \min \gamma R_{kj}^{F+}$, $\overline{\gamma R^{Fu}} = \max \gamma R_{kj}^{F+}$ and $\underline{\alpha J^{Tl}} = \min \alpha J_{kj}^{T-}$, $\overline{\alpha J^{Tl}} = \max \alpha J_{kj}^{T-}$, $\underline{\alpha J^{Tu}} = \min \alpha J_{kj}^{T+}$, $\overline{\alpha J^{Tu}} = \max \alpha J_{kj}^{T+}$, $\underline{\beta J^{Il}} = \min \beta J_{kj}^{I-}$, $\overline{\beta J^{Il}} = \max \beta J_{kj}^{I-}$, $\underline{\beta J^{Iu}} = \min \beta J_{kj}^{I+}$, $\overline{\beta J^{Iu}} = \max \beta J_{kj}^{I+}$, $\underline{\gamma J^{Fl}} = \min \gamma J_{kj}^{F-}$, $\overline{\gamma J^{Fl}} = \max \gamma J_{kj}^{F-}$, $\underline{\gamma J^{Fu}} = \min \gamma J_{kj}^{F+}$, $\overline{\gamma J^{Fu}} = \max \gamma J_{kj}^{F+}$.

Then,

$$\begin{array}{l} \left\langle (\underline{\vartheta}, \underline{\tau}); [\underline{\alpha R^{Tl} e^{i2\pi(\alpha J^{Tl})}}, \underline{\alpha R^{Tu} e^{i2\pi(\alpha J^{Tu})}}], [\underline{\beta R^{Il} e^{i2\pi(\beta J^{Il})}}, \underline{\beta R^{Iu} e^{i2\pi(\beta J^{Iu})}}], [\overline{\gamma R^{Fl} e^{i2\pi(\gamma J^{Fl})}}, \overline{\gamma R^{Fu} e^{i2\pi(\gamma J^{Fu})}}] \right\rangle \\ \leq q - CDNNIVWA(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \dots, \widetilde{\Phi}_m) \leq \\ \left\langle (\overline{\vartheta}, \overline{\tau}); [\overline{\alpha R^{Tl} e^{i2\pi(\alpha J^{Tl})}}, \overline{\alpha R^{Tu} e^{i2\pi(\alpha J^{Tu})}}], [\overline{\beta R^{Il} e^{i2\pi(\beta J^{Il})}}, \overline{\beta R^{Iu} e^{i2\pi(\beta J^{Iu})}}], [\underline{\gamma R^{Fl} e^{i2\pi(\gamma J^{Fl})}}, \underline{\gamma R^{Fu} e^{i2\pi(\gamma J^{Fu})}}] \right\rangle. \end{array}$$

(Boundedness property).

Proof. The proof of Theorem 4.3 is proposed in Appendix C. □

Theorem 4.4. *Let* $\widetilde{\Phi}_k = \left\langle (\vartheta_{t_{kj}}, \tau_{t_{kj}}); [R^{Tl}_{t_{kj}} e^{i2\pi J^{Tl}_{t_{kj}}}, R^{Tu}_{t_{kj}} e^{i2\pi J^{Tu}_{t_{kj}}}], [R^{Il}_{t_{kj}} e^{i2\pi J^{Il}_{t_{kj}}}, R^{Iu}_{t_{kj}} e^{i2\pi J^{Iu}_{t_{kj}}}], [R^{Fl}_{t_{kj}} e^{i2\pi J^{Fl}_{t_{kj}}}, R^{Fu}_{t_{kj}} e^{i2\pi J^{Fu}_{t_{kj}}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ *and* $\widetilde{W}_k = \left\langle (\vartheta_{h_{kj}}, \tau_{h_{kj}}); [R^{Tl}_{h_{kj}} e^{i2\pi J^{Tl}_{h_{kj}}}, R^{Tu}_{h_{kj}} e^{i2\pi J^{Tu}_{h_{kj}}}], [R^{Il}_{h_{kj}} e^{i2\pi J^{Il}_{h_{kj}}}, R^{Iu}_{h_{kj}} e^{i2\pi J^{Iu}_{h_{kj}}}], [R^{Fl}_{h_{kj}} e^{i2\pi J^{Fl}_{h_{kj}}}, R^{Fu}_{h_{kj}} e^{i2\pi J^{Fu}_{h_{kj}}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \ldots, m); (j = 1, 2, \ldots, k_j)$ *be the families of q-CDNNIVWAs. For any i,* $\vartheta_{t_{kj}} \le \tau_{h_{kj}}$, $\left(\alpha R^{Tl}_{t_{kj}}\right)^2 + \left(\alpha R^{Tu}_{t_{kj}}\right)^2 \le \left(\alpha R^{Tl}_{h_{kj}}\right)^2 + \left(\alpha R^{Tu}_{h_{kj}}\right)^2$ *and* $\left(\beta R^{Il}_{t_{kj}}\right)^2 + \left(\beta R^{Iu}_{t_{kj}}\right)^2 \le \left(\beta R^{Il}_{h_{kj}}\right)^2 + \left(\beta R^{Iu}_{h_{kj}}\right)^2$ *and* $\left(\gamma R^{Fl}_{t_{kj}}\right)^2 + \left(\gamma R^{Fu}_{t_{kj}}\right)^2 \ge \left(\gamma R^{Fl}_{h_{kj}}\right)^2 + \left(\gamma R^{Fu}_{h_{kj}}\right)^2$ *and* $\left(\alpha J^{Tl}_{t_{kj}}\right)^2 + \left(\alpha J^{Tu}_{t_{kj}}\right)^2 \le \left(\alpha J^{Tl}_{h_{kj}}\right)^2 + \left(\alpha J^{Tu}_{h_{kj}}\right)^2$ *and* $\left(\beta J^{Il}_{t_{kj}}\right)^2 + \left(\beta J^{Iu}_{t_{kj}}\right)^2 \le \left(\beta J^{Il}_{h_{kj}}\right)^2 + \left(\beta J^{Iu}_{h_{kj}}\right)^2$ *and* $\left(\gamma J^{Fl}_{t_{kj}}\right)^2 + \left(\gamma J^{Fu}_{t_{kj}}\right)^2 \ge \left(\gamma J^{Fl}_{h_{kj}}\right)^2 + \left(\gamma J^{Fu}_{h_{kj}}\right)^2$ *or* $\widetilde{\Phi}_k \le \widetilde{W}_k$. *Then q-CDNNIVWA* $\left(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m\right) \le q - CDNNIVWA\left(\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_m\right)$. *(Monotonicity property)*

Proof. The proof of Theorem 4.4 is proposed in Appendix D. □

4.2. q-CDNNIVWG

Each input is raised to the power of its weight using the WG operator, which then multiplies the results to aggregate values.

Definition 4.2. Let

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R^{Tl}_k e^{i2\pi J^{Tl}_k}, R^{Tu}_k e^{i2\pi J^{Tu}_k}], [R^{Il}_k e^{i2\pi J^{Il}_k}, R^{Iu}_k e^{i2\pi J^{Iu}_k}], \\ [R^{Fl}_k e^{i2\pi J^{Fl}_k}, R^{Fu}_k e^{i2\pi J^{Fu}_k}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. The q-CDNNIVWG operator is defined as q-CDNNIVWG $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \odot_{k=1}^m \widetilde{\Phi}_k^{\wp_k}$.

Theorem 4.5. *Let*

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R^{Tl}_k e^{i2\pi J^{Tl}_k}, R^{Tu}_k e^{i2\pi J^{Tu}_k}], [R^{Il}_k e^{i2\pi J^{Il}_k}, R^{Iu}_k e^{i2\pi J^{Iu}_k}], \\ [R^{Fl}_k e^{i2\pi J^{Fl}_k}, R^{Fu}_k e^{i2\pi J^{Fu}_k}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. Then

$$q - CDNNIVWG(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) =$$
$$\left[\begin{array}{c} \left(\odot_{k=1}^m \vartheta_k^{\wp_k}, \odot_{k=1}^m \tau_k^{\wp_k}\right); \\ \left[\odot_{k=1}^m (\alpha R^{Tl}_k)^{\wp_k} e^{i2\pi \odot_{k=1}^m \left(\frac{\alpha J^{Tl}_k}{2\pi}\right)^{\xi_k}}, \odot_{k=1}^m (\alpha R^{Tu}_k)^{\wp_k}; e^{i2\pi \odot_{k=1}^m \left(\frac{\alpha J^{Tu}_k}{2\pi}\right)^{\xi_k}}\right]; \\ \left[\sqrt[q]{1 - \odot_{k=1}^m \left(1 - (\beta R^{Il}_k)^q\right)^{\wp_k}} e^{i2\pi \sqrt[q]{1 - \odot_{k=1}^m \left(1 - \left(\frac{\beta J^{Il}_k}{2\pi}\right)^q\right)^{\xi_k}}}, \right. \\ \left. \sqrt[q]{1 - \odot_{k=1}^m \left(1 - (\beta R^{Iu}_k)^q\right)^{\wp_k}} e^{i2\pi \sqrt[q]{1 - \odot_{k=1}^m \left(1 - \left(\frac{\beta J^{Iu}_k}{2\pi}\right)^q\right)^{\xi_k}}} \right]; \\ \left[\sqrt[2q]{1 - \odot_{k=1}^m \left(1 - (\gamma R^{Fl}_k)^{2q}\right)^{\wp_k}} e^{i2\pi \sqrt[2q]{1 - \odot_{k=1}^m \left(1 - \left(\frac{\gamma J^{Fl}_k}{2\pi}\right)^{2q}\right)^{\xi_k}}}, \right. \\ \left. \sqrt[2q]{1 - \odot_{k=1}^m \left(1 - (\gamma R^{Fu}_k)^{2q}\right)^{\wp_k}} e^{i2\pi \sqrt[2q]{1 - \odot_{k=1}^m \left(1 - \left(\frac{\gamma J^{Fu}_k}{2\pi}\right)^{2q}\right)^{\xi_k}}} \right]; \end{array} \right].$$

Proof. The proof follows from Theorem 4.1. □

Theorem 4.6. *If all* $\widetilde{\Phi}_k = \left\langle (\vartheta_k, \tau_k); [R^{Tl}_k e^{i2\pi J^{Tl}_k}, R^{Tu}_k e^{i2\pi J^{Tu}_k}], [R^{Il}_k e^{i2\pi J^{Il}_k}, R^{Iu}_k e^{i2\pi J^{Iu}_k}], [R^{Fl}_k e^{i2\pi J^{Fl}_k}, R^{Fu}_k e^{i2\pi J^{Fu}_k}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \ldots, m)$ *are equal with* $\widetilde{\Phi}_k = \widetilde{\Phi}$. *Then q-CDNNIVWG*$(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \widetilde{\Phi}$.

Proof. The proof follows from Theorem 4.2. □

The q-CDNNIVWG operator indicates that we can meet the monotonicity and boundedness conditions.

4.3. Generalized q-CDNNIVWA (q-GCDNNIVWA)

In order to calculate the WA and WG using AOs, we establish the following operating regulations for the q-CDNNIVS.

Definition 4.3. Let

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R^{Tl}_k e^{i2\pi J^{Tl}_k}, R^{Tu}_k e^{i2\pi J^{Tu}_k}], [R^{Il}_k e^{i2\pi J^{Il}_k}, R^{Iu}_k e^{i2\pi J^{Iu}_k}], \\ [R^{Fl}_k e^{i2\pi J^{Fl}_k}, R^{Fu}_k e^{i2\pi J^{Fu}_k}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVN. Then q-GCDNNIVWA $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \left(\bigoplus_{k=1}^m \wp_k \widetilde{\Phi}_k^q\right)^{1/q}$ is called a q-GCDNNIVWA operator.

Theorem 4.7. *Let*

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], \\ {[R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}]}, [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. Then q-GCDNNIVWA $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) =$

$$\left[\begin{array}{c}
\left(\left(\bigoplus_{k=1}^m \wp_k \vartheta_k^q\right)^{1/q}, \left(\bigoplus_{k=1}^m \wp_k \tau_k^q\right)^{1/q}\right);\\
\left[\begin{array}{l}
\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left((\alpha R_k^{Tl})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}},\\
\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left((\alpha R_k^{Tu})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}}
\end{array}\right],\\
\left[\begin{array}{l}
\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left((\beta R_k^{Il})^q\right)^{q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{1/q}},\\
\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left((\beta R_k^{Iu})^q\right)^{q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{1/q}}
\end{array}\right],\\
\left[\begin{array}{l}
\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k}\right)^{2q}\right)}^{\,1/q}\\
e^{i2\pi\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^{2q}\right)}^{\,1/q}},\\
\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fu})^{2q}\right)^q}\right)^{\wp_k}\right)^{2q}\right)}^{\,1/q}\\
e^{i2\pi\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^{2q}\right)}^{\,1/q}},
\end{array}\right]
\end{array}\right].$$

Proof. The proof of Theorem 4.7 is proposed in Appendix E. □

Remark 4.1. q-GCDNNIVWA is changed to q-CDNNIVWA whenever $q = 1$.

Theorem 4.8. *If all* $\widetilde{\Phi}_k = \left\langle (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], [R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \ldots, m)$ *are equal with* $\widetilde{\Phi}_k = \widetilde{\Phi}$. *Then* q-GCDNNIVWA$(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \widetilde{\Phi}$.

Proof. Proof follows from Theorem 4.2. □

Remark 4.2. We satisfy the monotonicity and boundedness properties of the q-GCDNNIVWA.

4.4. Generalized q-CDNNIVWG (q-GCDNNIVWG)

The GWG operator combines the power transformation capabilities of generalized operators with the multiplicative basis of WG.

Definition 4.4. Let

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], \\ {[R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}]}, [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. Then q-GCDNNIVWG $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \frac{1}{q}\left(\odot_{k=1}^m (q\widetilde{\Phi}_k)^{\wp_k}\right)$.

Theorem 4.9. *Let*

$$\widetilde{\Phi}_k = \left\langle \begin{array}{l} (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], \\ {[R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}]}, [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

be a collection of q-CDNNIVNs. Then q-GCDNNIVWG$(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m)$ *is defined as follows:*

$$\left[\begin{array}{c} \left(\frac{1}{q} \odot_{k=1}^m (q\vartheta_k)^{\wp_k}, \frac{1}{q} \odot_{k=1}^m (q\tau_k)^{\wp_k} \right); \\ \left[\begin{array}{l} \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-(\alpha R_k^{Tl})^{2q}\right)^q}\right)^{\wp_k}\right)^{2q}\right)}^{\,1/q} e^{i2\pi \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^{2q}\right)}^{\,1/q}}, \\ \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-(\alpha R_k^{Tu})^{2q}\right)^q}\right)^{\wp_k}\right)^{2q}\right)}^{\,1/q} e^{i2\pi \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^m\left(\sqrt[2q]{1-\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^{2q}\right)}^{\,1/q}} \end{array} \right], \\ \left[\begin{array}{l} \left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left((\beta R_k^{Il})^q\right)^q\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^q\right)^{\xi_k}}\right)^{1/q}}, \\ \left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left((\beta R_k^{Iu})^q\right)^q\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^q\right)^{\xi_k}}\right)^{1/q}} \end{array} \right] \\ \left[\begin{array}{l} \left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left((\gamma R_k^{Fl})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}}, \\ \left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left((\gamma R_k^{Fu})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}} \end{array} \right] \end{array} \right].$$

Remark 4.3. q-GCDNNIVWG operator is changed to q-CDNNIVWG operator whenever $q = 1$.

Remark 4.4. The q-GCDNNIVWG operator satisfies the monotonicity and boundness requirements.

Theorem 4.10. *If all* $\widetilde{\Phi}_k = \left\langle (\vartheta_k, \tau_k); [R_k^{Tl} e^{i2\pi J_k^{Tl}}, R_k^{Tu} e^{i2\pi J_k^{Tu}}], [R_k^{Il} e^{i2\pi J_k^{Il}}, R_k^{Iu} e^{i2\pi J_k^{Iu}}], [R_k^{Fl} e^{i2\pi J_k^{Fl}}, R_k^{Fu} e^{i2\pi J_k^{Fu}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$ $(k = 1, 2, \ldots, m)$ *are equal with* $\widetilde{\Phi}_k = \widetilde{\Phi}$. *Then*

$$q - GCDNNIVWG(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) = \widetilde{\Phi}.$$

4.5. Scientific analysis and advantages of the proposed AOs

The AOs proposed in Sections 4.1–4.4 play an important role in combining complex uncertain information in the q-CDNNIVS environment. These operators are not only mathematical tools. They also provide a scientific basis for integrating expert evaluations amid vagueness, inconsistency, indeterminacy, and periodic uncertainty. In practical MCDM problems, each alternative is evaluated through several criteria. These criteria may not contribute equally to the final decision. Therefore, suitable AOs are required to combine all criterion values into a single representative assessment without losing important information.

The q-CDNNIVWA operator proposed in Section 4.1 offers a balanced aggregation of criteria. It is useful when a decision-maker allows compensation among criteria. For example, in industrial robot selection, a robot with moderate speed but strong flexibility and payload may still be acceptable if the overall weighted performance is high. This operator provides a smooth collective assessment by accounting for each attribute's relative importance. It is suitable for decision problems where the final ranking should reflect the overall contribution of all criteria.

The q-CDNNIVWG operator proposed in Section 4.2 exhibits more conservative decision-making behavior. It is useful when poor performance in one criterion should not be fully compensated by strong performance in another criterion. This is important in engineering applications, where a weak attribute may render an alternative less practical. For example, an industrial robot with high payload but low standardization or poor flexibility may not be the best option in a productive system. The geometric structure of this operator makes it more sensitive to low-performing criteria. Therefore, it supports risk-aware and consistency-oriented decision-making.

The q-GCDNNIVWA operator proposed in Section 4.3 extends the basic WA operator by introducing a generalized form. Its main advantage is flexibility. It allows the decision-maker to adjust the aggregation behavior according to the decision context. When the generalization parameter

Table 2
Scientific role and advantages of the proposed AOs.

Operator	Main aggregation behavior	Scientific advantage	Practical decision-making benefit
q-CDNNIVWA	Weighted averaging	Combines all criteria in a balanced and compensatory way	Suitable when strong performance in some criteria can compensate for moderate performance in others
q-CDNNIVWG	Weighted geometric aggregation	Gives greater influence to weak criterion performance	Suitable for conservative and risk-aware selection problems
q-GCDNNIVWA	Generalized weighted averaging	Provides an adjustable compensatory aggregation structure	Useful when the decision-maker wants flexible ranking behavior under different uncertainty levels
q-GCDNNIVWG	Generalized weighted geometric aggregation	Provides an adjustable conservative aggregation structure	Useful for robustness checking and sensitivity analysis

takes a specific value, this operator reduces to the basic q-CDNNIVWA operator. Thus, it includes the WA operator as a special case. This property makes the operator more adaptable, as it can present both ordinary averaging and more flexible aggregation under different levels of uncertainty.

The q-GCDNNIVWG operator proposed in Section 4.4 extends the basic WG operator. It is useful when the decision-maker needs a conservative but adjustable aggregation mechanism. Similar to the q-GCDNNIVWA operator, it includes the q-CDNNIVWG operator as a special case under a specific parameter condition. This feature allows the method to evaluate the stability of rankings under different aggregation attitudes. Therefore, this operator is valuable for sensitivity analysis and robustness checking in uncertain decision-making problems.

From a scientific perspective, all four operators preserve the internal structure of q-CDNNIV information. They aggregate truth, indeterminacy, and falsity degrees separately while also maintaining interval-valued information, normal fuzzy parameters, amplitude terms, phase terms, and reference parameters. This is important because these components describe different aspects of uncertainty. The truth degree reflects the level of support for an alternative. The indeterminacy degree reflects incomplete or unclear information. The degree of falsity reflects the extent of opposition to the alternative. The interval-valued form represents lower and upper assessment bounds. The normal fuzzy component captures central tendency and dispersion. The complex phase term helps represent periodic or time-dependent behavior. The reference parameters provide additional flexibility in controlling the admissible information space.

The algebraic properties of the proposed operators also have direct implications for decision-making. Idempotency ensures that if all input assessments are the same, the aggregated result remains unchanged. This prevents artificial distortion. Boundedness ensures that the aggregated value remains within the range of the input information. This prevents unrealistic results. Monotonicity ensures that improvement in input assessments does not reduce the final aggregated value. This supports rational comparison among alternatives. Commutativity ensures that the order of criteria does not affect the aggregation result when the same weights are used. Associativity supports consistent stepwise aggregation.

Therefore, the proposed operators provide both mathematical validity and practical usefulness. The q-CDNNIVWA operator is suitable for balanced evaluation. The q-CDNNIVWG operator is suitable for conservative evaluation. The q-GCDNNIVWA operator provides flexible compensatory aggregation. The q-GCDNNIVWG operator provides flexible conservative aggregation. Together, these operators offer a robust decision-making framework for complex MCDM problems, especially when alternatives must be evaluated under uncertain, interval-valued, neutrosophic, and periodic information.

The comparative advantages of these operators are summarized in Table 2.

5. MADM approach based on q-CDNNIVNs

This section presents a strategy for addressing the MADM problem within the IVNS framework, utilizing similarity measures to evaluate alternatives and apply them to q-CDNNIVNs. In MADM situations, the ideal point concept has been used to identify the optimal choice among available options. Using data from decision-makers, an MADM algorithm selects the best options based on weighted qualities. Examine a MADM problem that has n characteristics and m possibilities. Let $C = \{U_1, U_2, \dots, U_n\}$ be the set of attributes and $\Phi = \{\Phi_1, \Phi_2, \dots, \Phi_m\}$ be a discrete collection of alternatives. Ratings tell us how well option Φ_m performs in relation to attribute U_n. Assume further that the decision makers allocated the weight vector $w = \{\wp_1, \wp_2, \dots, \wp_n\}$ to the attributes $C = \{U_1, U_2, \dots, U_n\}$. Information on the values associated with MADM choices can be found in the decision matrix that follows. The next steps use the suggested operators to solve this DM problem.

$$\widetilde{\Phi}_{kj} = \left\langle (\vartheta_{kj}, \tau_{kj}); [R^{Tl}_{kj} e^{i2\pi J^{Tl}_{kj}}, R^{Tu}_{kj} e^{i2\pi J^{Tu}_{kj}}], [R^{Il}_{kj} e^{i2\pi J^{Il}_{kj}}, R^{Iu}_{kj} e^{i2\pi J^{Iu}_{kj}}], [R^{Fl}_{kj} e^{i2\pi J^{Fl}_{kj}}, R^{Fu}_{kj} e^{i2\pi J^{Fu}_{kj}}], [\alpha_k, \beta_k, \gamma_k] \right\rangle$$

denote q-CDNNIVN of alternative Φ_k in attribute U_j. Since $[R^{Tl}_{kj} e^{i2\pi J^{Tl}_{kj}}, R^{Tu}_{kj} e^{i2\pi J^{Tu}_{kj}}], [R^{Il}_{kj} e^{i2\pi J^{Il}_{kj}}, R^{Iu}_{kj} e^{i2\pi J^{Iu}_{kj}}]$, $[R^{Fl}_{kj} e^{i2\pi J^{Fl}_{kj}}, R^{Fu}_{kj} e^{i2\pi J^{Fu}_{kj}}]$, $\alpha_k, \beta_k, \gamma_k \in [0,1]$ and $0 \le (\alpha R^{Tu}_{kj}(x))^q + (\beta R^{Iu}_{kj}(x))^q + (\gamma R^{Fu}_{kj}(x))^q \le 1$ and $0 \le (\alpha J^{Tu}_{kj}(x))^q + (\beta J^{Iu}_{kj}(x))^q + (\gamma J^{Fu}_{kj}(x))^q \le 1$. The decision matrix $\mathscr{D} = (\widetilde{\Phi}_{kj})_{m\times p}$, for $k = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$.

Fig. 3 shows an algorithm for the suggested models using different operators.

The steps suggested by the algorithm are as follows.

Step 1: Create the set of decision values and reference parameters for q-CDNNIVS.
Step 2: Find the normalized decision values, from $\mathscr{D} = (\widetilde{\Phi}_{kj})_{m\times p}$ to $\overrightarrow{\mathscr{D}} = (\Phi_{kj})_{m\times p}$; where

$$\Phi_{kj} = \left\langle \begin{array}{l} (\overrightarrow{\vartheta_{kj}}, \overrightarrow{\tau_{kj}}); [\overrightarrow{R^{Tl}_{kj} e^{ij2\pi J^{Tl}_{kj}}}, \overrightarrow{R^{Tu}_{kj} e^{ij2\pi J^{Tu}_{kj}}}], [\overrightarrow{R^{Il}_{kj} e^{ij2\pi J^{Il}_{kj}}}, \overrightarrow{R^{Iu}_{kj} e^{ij2\pi J^{Iu}_{kj}}}], \\ [\overrightarrow{R^{Fl}_{kj} e^{ij2\pi J^{Fl}_{kj}}}, \overrightarrow{R^{Fu}_{kj} e^{ij2\pi J^{Fu}_{kj}}}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

and

$$\overrightarrow{\vartheta_{kj}} = \frac{\vartheta_{kj}}{\max_k(\vartheta_{kj})}, \overrightarrow{\tau_{kj}} = \frac{\tau_{kj}}{\max_k(\tau_{kj})} \cdot \frac{\tau_{kj}}{\vartheta_{kj}}$$

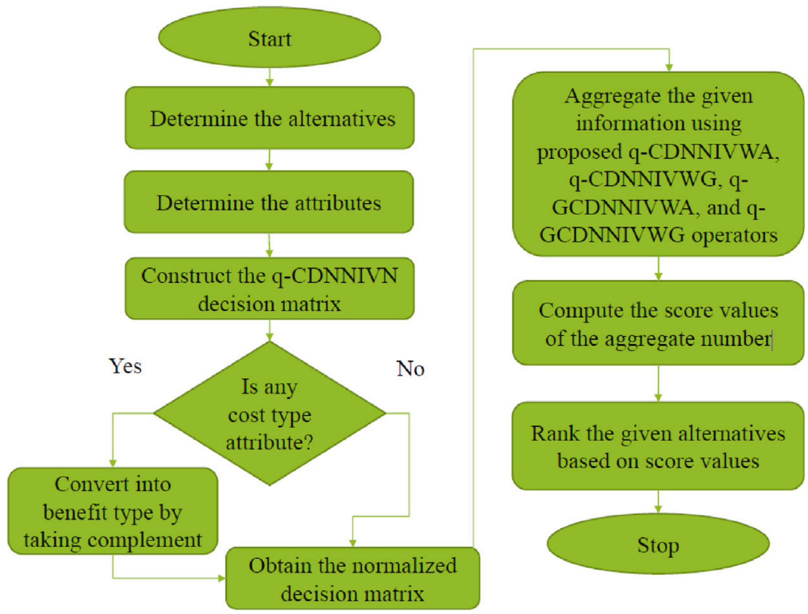

Fig. 3. Algorithm of the suggested models.

$$\overrightarrow{\alpha R^{Tl}_{kj}} = \alpha R^{Tl}_{kj}, \overrightarrow{\alpha R^{Tu}_{kj}} = \alpha R^{Tu}_{kj}, \overrightarrow{\alpha R^{Il}_{kj}} = \beta R^{Il}_{kj}, \overrightarrow{\beta R^{Iu}_{kj}} = \beta R^{Iu}_{kj}, \overrightarrow{\gamma R^{Fl}_{kj}} = \gamma R^{Fl}_{kj}, \overrightarrow{\gamma R^{Fu}_{kj}} = \gamma R^{Fu}_{kj}.$$

Step 3: Find the aggregate values using these operators. For q-CDNNIVS, attribute k_j in $\widetilde{\Phi}_k$,

$$\Phi_{kj} = \left\langle \begin{array}{l} (\overrightarrow{\vartheta_{kj}}, \overrightarrow{\tau_{kj}}); [\overrightarrow{R^{Tl}_{kj} e^{ij2\pi J^{Tl}_{kj}}}, \overrightarrow{R^{Tu}_{kj} e^{ij2\pi J^{Tu}_{kj}}}], [\overrightarrow{R^{Il}_{kj} e^{ij2\pi J^{Il}_{kj}}}, \overrightarrow{R^{Iu}_{kj} e^{ij2\pi J^{Iu}_{kj}}}], \\ [\overrightarrow{R^{Fl}_{kj} e^{ij2\pi J^{Fl}_{kj}}}, \overrightarrow{R^{Fu}_{kj} e^{ij2\pi J^{Fu}_{kj}}}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

is aggregated into

$$\Phi_{k} = \left\langle \begin{array}{l} (\overrightarrow{\vartheta_{k}}, \overrightarrow{\tau_{k}}); [\overrightarrow{R^{Tl}_{k} e^{i2\pi J^{Tl}_{k}}}, \overrightarrow{R^{Tu}_{k} e^{i2\pi J^{Tu}_{k}}}], [\overrightarrow{R^{Il}_{k} e^{i2\pi J^{Il}_{k}}}, \overrightarrow{R^{Iu}_{k} e^{i2\pi J^{Iu}_{k}}}], \\ [\overrightarrow{R^{Fl}_{k} e^{i2\pi J^{Fl}_{k}}}, \overrightarrow{R^{Fu}_{k} e^{i2\pi J^{Fu}_{k}}}], [\alpha_k, \beta_k, \gamma_k] \end{array} \right\rangle$$

Step 4: Find the score and accuracy values are as follows:

$$S_1(\widetilde{\Phi}) = \frac{\vartheta}{2}(C+D), S_2(\widetilde{\Phi}) = \frac{\tau}{2}(C+D), \mathbb{S}(\widetilde{\Phi}) = \frac{S_1(\widetilde{\Phi}) + S_2(\widetilde{\Phi})}{2}$$

where

$$C = \left(\frac{(\alpha R^{Tl}_k)^2 + (\alpha R^{Tu}_k)^2}{2} - \frac{(\beta R^{Il}_k)^2 + (\beta R^{Iu}_k)^2}{2} + 2 - \frac{(\gamma R^{Fl}_k)^2 + (\gamma R^{Fu}_k)^2}{2} \right)$$

$$D = \left(\frac{(\alpha J^{Tl}_k)^2 + (\alpha J^{Tu}_k)^2}{2} - \frac{(\beta J^{Il}_k)^2 + (\beta J^{Iu}_k)^2}{2} - \frac{(\gamma J^{Fl}_k)^2 + (\gamma J^{Fu}_k)^2}{2} \right),$$

where $\mathbb{S}(\widetilde{\Phi}) \in [-1, 1]$.
Find the accuracy values are as follows:

$$H_1(\widetilde{\Phi}) = \frac{\vartheta}{2}(C_1 + D_1), H_2(\widetilde{\Phi}) = \frac{\tau}{2}(C_1 + D_1), \mathbb{H}(\widetilde{\Phi}) = \frac{H_1(\widetilde{\Phi}) + H_2(\widetilde{\Phi})}{2}.$$

where

$$C_1 = \left(\frac{(\alpha R^{Tl}_k)^2 + (\alpha R^{Tu}_k)^2}{2} + \frac{(\beta R^{Il}_k)^2 + (\beta R^{Iu}_k)^2}{2} + \frac{(\gamma R^{Fl}_k)^2 + (\gamma R^{Fu}_k)^2}{2} \right)$$

$$D_1 = \left(\frac{(\alpha J^{Tl}_k)^2 + (\alpha J^{Tu}_k)^2}{2} + \frac{(\beta J^{Il}_k)^2 + (\beta J^{Iu}_k)^2}{2} + \frac{(\gamma J^{Fl}_k)^2 + (\gamma J^{Fu}_k)^2}{2} \right),$$

where $\mathbb{H}(\widetilde{\Phi}) \in [0, 1]$.

Step 5: A decision that has the greatest score is the best or most perfect. Based on their score values, the other choices are likewise arranged in decreasing order. Given that the result shows that $\max \mathscr{D}^*_k$ is the optimal value, this is the best way to solve the current problem.

The following example demonstrates the practical application of the algorithm. The problem is examined to illustrate how the proposed strategy can be effectively utilized in a real-world scenario:

6. Case study of industrial robot selection

6.1. Problem description

Industrial robots have transformed manufacturing and production processes across a range of industries, demonstrating their ability to improve efficiency, accuracy, and security. These robotic technologies, which enable more effective operations and lower human error, are essential to modern industrial practices. Industrial robots are no longer just used in traditional automobile lines. Robots are used in machining, packing, inspection, and logistics cells within the same factory in modern robotic automation projects. Industrial robot applications can be categorized into pattern handling, joining, surface processing, quality control, and specific roles in food, medicine, and energy due to the large number of applications. For engineers and managers, robotics becomes a flexible toolbox rather than a single technology. Several robot types are matched to certain tasks based on payload, reach, and accuracy. To further explain the earlier-mentioned DM process, we now examine a real-world example. The best industrial robot selection problem, which is regarded as an MADM problem, was chosen in this section using the suggested weighted operators. Robots are being used by a variety of manufacturers worldwide to automate tasks, increase worker safety, reduce waste, and lower operating expenses simultaneously. A growing number of industrial robots have been used in manufacturing environments in recent years, leading to increased demand for models that fit a wide range of specific industries and applications. With numerous industrial robots available on the market, this study will examine their distinctive capabilities, working principles, and the industries in which they are most commonly utilized.

6.1.1. Definition of alternatives

In decision-making, alternatives are the different options or solutions a DM can consider when evaluating a given problem. The process of defining alternatives involves identifying a range of viable choices that address the criteria set for the decision. The effectiveness of the decision-making process relies on how well these alternatives are characterized and assessed. Each alternative is evaluated based on multiple factors, which help determine the most suitable course of action.

(1) Cartesian Robots (A)**:** Cartesian robots are sometimes called linear robots or gantry robots because they use Cartesian coordinates to move in straight lines on three axes (up and down, in and out, and side to side). The highly flexible configuration of Cartesian robots allows users to adjust speed, precision, stroke length, and size to suit their needs. CNC machines and 3D printers commonly use Cartesian Robots in industrial applications. In general, servomotor drives are used to drive Cartesian robots. Linear actuators may take different forms depending on their specific application. There are several types of drive systems, including belts, cables, screws, pneumatics, rack-and-pinions, and linear motors. It is possible to order complete Cartesian robots from manufacturers that do not require modifications. Depending on the user's needs, some manufacturers offer components as modules. A Cartesian robot is much easier to program than a six-axis robot. Cartesian robots can be controlled using a single motion controller. Robots are easy to control owing to their linear motion. Cartesian robots can be controlled without complex arrays of microchips and PLCs. Similarly, the robot's movement can be programmed more easily owing to this attribute.

(2) Articulated Robots (B)**:** Articulated robots closely resemble human arms in their mechanical movement and configuration. The arms mounted on the bases had twisting joints. From two to ten rotator joints, the arm can be configured with up to ten axes, each of which allows for greater degrees of movement. Typically, most articulating robots have four or six axes. Assembling, welding, manipulating materials, caring for machines, and packaging are some of the many uses of articulated robots. Because of their incredible versatility, articulating robots are widely used. Other robot structures may be limited by design or axis restraints, whereas articulated robots can automate most production tasks. Automation of material removal applications using articulated robots is also possible. Several processes are involved, such as drilling, sanding, grinding, polishing, and buffing. Materials removal applications include laser, plasma, waterjet, and ultrasonic cutting, all of which fall under this category. Owing to the application-specific nature of tools, end-of-arm tooling varies depending on the material that needs to be removed.

(3) Cylindrical Robots (C)**:** In a cylindrical robot, the base is rotary, whereas prismatic joints connect the links. An expendable arm that moves vertically and rotates around the rotating shaft provides the robots with a cylindrical work envelope. Cpact designs make cylindrical robots ideal for applications such as simple assembly, machine maintenance, and coatings in tight spaces. To indicate point positions, cylindrical robots use a three-dimensional coordinate system based on distances from a reference axis. In addition, the axis position is determined by the distance from the axis perpendicular to a selected reference plane and its distance from the reference direction. A robot with this type of rotational symmetry can help distribute heat, generate electromagnetic fields, and secure rounded cross-sections in pipework. It is also common for cylindrical robots to be fast. However, this speed can cause rotational inertia issues that affect repeatability if not correctly configured.

(4) Delta Robots (D)**:** Delta Robots are characterized by three arms mounted on a base above the work area. The Delta Robot can move precisely and sensitively at high speeds thanks to its dome-shaped end effector, which is directly controlled by all three arms. Delta robots are widely utilized for fast pick-and-place operations, particularly in industries such as electronics, food processing, and pharmaceuticals. High acceleration and speed are two characteristics of delta robots. Unlike other types of industrial robots, these are designed to be faster. Robot arms with motors are typically designed for heavy payload applications, making them ideal for industrial robotics. This adds weight to the robot arm and limits its speed due to the arm's motors. In delta robots, all the motors are mounted above the work area. Thus, the robot keeps its main weight stationary. The lightweight robot arms provide whole movement, enabling extremely high acceleration and speed.

(5) Polar Robots (E)**:** A spherical or polar robot is designed with an arm featuring two rotary joints and one linear joint, which are connected to a base through a twisting joint. The robot operates within a spherical work envelope defined by the arrangement of its axes, which form a polar coordinate system. This configuration enables the robot to move efficiently within its defined workspace, allowing it to perform a range of tasks across various industrial applications. Material handling, die casting, injection molding, and welding are among the frequent applications for polar robots. It is estimated that machines with only three axes can significantly outperform human and manual labor in production. This has led automobile and industrial manufacturers to seek them out. As a result of this rush to adopt this new technology, the industrial robot revolution began, and 6-axis robots and many other improvements in robotics emerged.

Table 3
Decision values with 1st attribute.

	U_1
A	$\langle (0.8, 0.75); [0.9e^{i2\pi(0.8)}, 0.95e^{i2\pi(0.85)}], [0.65e^{i2\pi(0.65)}, 0.7e^{i2\pi(0.7)}], [0.65e^{i2\pi(0.65)}, 0.7e^{i2\pi(0.75)}], [0.55, 0.6, 0.5] \rangle$
B	$\langle (0.7, 0.55); [0.8e^{i2\pi(0.85)}, 0.85e^{i2\pi(0.9)}], [0.7e^{i2\pi(0.7)}, 0.75e^{i2\pi(0.8)}], [0.55e^{i2\pi(0.55)}, 0.6e^{i2\pi(0.6)}], [0.4, 0.5, 0.65] \rangle$
C	$\langle (0.85, 0.8); [0.7e^{i2\pi(0.75)}, 0.75e^{i2\pi(0.8)}], [0.45e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.65)}], [0.7e^{i2\pi(0.7)}, 0.75e^{i2\pi(0.75)}], [0.5, 0.55, 0.6] \rangle$
D	$\langle (0.6, 0.45); [0.85e^{i2\pi(0.8)}, 0.9e^{i2\pi(0.85)}], [0.6e^{i2\pi(0.45)}, 0.65e^{i2\pi(0.55)}], [0.65e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.85)}], [0.6, 0.45, 0.55] \rangle$
E	$\langle (0.9, 0.85); [0.75e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.75)}], [0.5e^{i2\pi(0.65)}, 0.55e^{i2\pi(0.7)}], [0.4e^{i2\pi(0.75)}, 0.55e^{i2\pi(0.8)}], [0.55, 0.5, 0.6] \rangle$

Table 4
Decision values with 2nd attribute.

	U_2
A	$\langle (0.8, 0.55); [0.85e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.75)}], [0.7e^{i2\pi(0.6)}, 0.85e^{i2\pi(0.75)}], [0.8e^{i2\pi(0.55)}, 0.85e^{i2\pi(0.65)}], [0.45, 0.5, 0.55] \rangle$
B	$\langle (0.85, 0.5); [0.65e^{i2\pi(0.65)}, 0.7e^{i2\pi(0.7)}], [0.6e^{i2\pi(0.65)}, 0.8e^{i2\pi(0.8)}], [0.6e^{i2\pi(0.65)}, 0.65e^{i2\pi(0.75)}], [0.7, 0.45, 0.35] \rangle$
C	$\langle (0.9, 0.7); [0.75e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.65)}], [0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.75)}], [0.55e^{i2\pi(0.75)}, 0.65e^{i2\pi(0.8)}], [0.65, 0.35, 0.3] \rangle$
D	$\langle (0.7, 0.6); [0.8e^{i2\pi(0.55)}, 0.85e^{i2\pi(0.6)}], [0.75e^{i2\pi(0.65)}, 0.85e^{i2\pi(0.7)}], [0.7e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.85)}], [0.7, 0.45, 0.4] \rangle$
E	$\langle (0.8, 0.7); [0.85e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.55)}], [0.75e^{i2\pi(0.45)}, 0.8e^{i2\pi(0.65)}], [0.75e^{i2\pi(0.8)}, 0.85e^{i2\pi(0.85)}], [0.6, 0.4, 0.55] \rangle$

Table 5
Decision values with 3rd attribute.

	U_3
A	$\langle (0.7, 0.6), [0.5, e^{i2\pi(0.7)}, 0.65, e^{i2\pi(0.75)}], [0.65, e^{i2\pi(0.55)}, 0.7, e^{i2\pi(0.7)}], [0.45, e^{i2\pi(0.65)}, 0.5, e^{i2\pi(0.8)}], [0.4, 0.6, 0.5] \rangle$
B	$\langle (0.55, 0.45), [0.7, e^{i2\pi(0.75)}, 0.75, e^{i2\pi(0.8)}], [0.7, e^{i2\pi(0.6)}, 0.75, e^{i2\pi(0.8)}], [0.5, e^{i2\pi(0.55)}, 0.6, e^{i2\pi(0.65)}], [0.35, 0.5, 0.4] \rangle$
C	$\langle (0.65, 0.5), [0.8, e^{i2\pi(0.65)}, 0.85, e^{i2\pi(0.7)}], [0.6, e^{i2\pi(0.7)}, 0.65, e^{i2\pi(0.85)}], [0.35, e^{i2\pi(0.45)}, 0.4, e^{i2\pi(0.7)}], [0.5, 0.55, 0.35] \rangle$
D	$\langle (0.75, 0.7), [0.6, e^{i2\pi(0.55)}, 0.65, e^{i2\pi(0.75)}], [0.55, e^{i2\pi(0.75)}, 0.65, e^{i2\pi(0.8)}], [0.6, e^{i2\pi(0.5)}, 0.65, e^{i2\pi(0.65)}], [0.4, 0.65, 0.5] \rangle$
E	$\langle (0.8, 0.65), [0.8, e^{i2\pi(0.45)}, 0.85, e^{i2\pi(0.65)}], [0.7, e^{i2\pi(0.65)}, 0.85, e^{i2\pi(0.7)}], [0.5, e^{i2\pi(0.7)}, 0.55, e^{i2\pi(0.85)}], [0.3, 0.4, 0.45] \rangle$

Table 6
Decision values with 4th attribute.

	U_4
A	$\langle (0.6, 0.55); [0.45e^{i2\pi(0.55)}, 0.55e^{i2\pi(0.65)}], [0.8e^{i2\pi(0.65)}, 0.85e^{i2\pi(0.7)}], [0.7e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.75)}], [0.35, 0.7, 0.5] \rangle$
B	$\langle (0.8, 0.75); [0.6e^{i2\pi(0.65)}, 0.65e^{i2\pi(0.75)}], [0.6e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.65)}], [0.6e^{i2\pi(0.65)}, 0.75e^{i2\pi(0.7)}], [0.4, 0.45, 0.35] \rangle$
C	$\langle (0.75, 0.7); [0.5e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.8)}], [0.55e^{i2\pi(0.7)}, 0.75e^{i2\pi(0.75)}], [0.65e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.55)}], [0.5, 0.4, 0.4] \rangle$
D	$\langle (0.85, 0.8); [0.65e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.85)}], [0.45e^{i2\pi(0.75)}, 0.65e^{i2\pi(0.8)}], [0.6e^{i2\pi(0.6)}, 0.8e^{i2\pi(0.65)}], [0.55, 0.5, 0.45] \rangle$
E	$\langle (0.8, 0.75); [0.7e^{i2\pi(0.75)}, 0.85e^{i2\pi(0.8)}], [0.65e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.85)}], [0.75e^{i2\pi(0.7)}, 0.85e^{i2\pi(0.75)}], [0.45, 0.65, 0.35] \rangle$

6.1.2. Definition of evaluation criteria

Robotics investments require careful consideration of all the relevant aspects of the application before a final decision is made. Furthermore, the following should be taken into account:

- **Reach and payload** (U_1) — As part of your robot selection process, you should prioritize reach and payload, as these criteria will immediately narrow the number of suitable options. Lightweight handling technologies would not be considered for a large, heavy load, for example. Meanwhile, a Cartesian robot with a low payload weight may be sufficient if its reach is long.
- **Flexibility** (U_2) — A robot with five or six degrees of freedom is the only solution for an application requiring five or six degrees of freedom. If one or two robot units are required at low cost, re-purposed (used) robot units might be considered. There is no need to spend more money on more axes than an application requires for simple applications, such as the positioning and loading of small parts, electronic part insertion, and loading boxes and machine tools.
- **Speed** (U_3) — Is the application requiring a high pick rate, such as the one provided by a delta robot, or would a lower pick rate, such as that of a Cartesian gantry or Polar Robot, suffice?
- **Standardization** (U_4)-It is not necessary to select the cheapest or most ideally suited robot to perform a task in a company or industry; however, standardization is a valid consideration for business reasons. There are times when the easy path (and riskiest path) is the best.

6.2. Problem solution

A random selection methodology was used to choose five distinct kinds of robots. The following elements should be considered before making a decision. When selecting robot systems and their weights, four considerations must be taken into account. Here $w = \{0.4, 0.3, 0.2, 0.1\}$ and $E = \{$reach payload, flexibility, speed and standardization$\}$. We look for the best solution that meets our needs to choose the best option from the available options.

Step 1. The evaluation values reported in Tables 3 to 6 are not directly assigned by experts in their final q-CDNNIV mathematical form. In practical decision-making applications, experts usually provide linguistic or interval-based assessments for each alternative with respect to each criterion. For example, they may describe the performance of an alternative as low, medium, high, very high, or provide an interval score based on their experience and available information.

These initial assessments are then transformed into q-CDNNIV numbers through predefined mapping rules and membership construction functions. The transformed q-CDNNIV values include truth degree, indeterminacy degree, falsity degree, interval-valued bounds, normal fuzzy information, and complex-valued phase information. Therefore, the apparent complexity of the values reflects the mathematical encoding of uncertain information rather than the complexity of expert input.

This procedure makes the proposed method practical for real applications. Experts are not required to manually construct complex q-CDNNIV values. They only provide intuitive linguistic or interval-based judgments, while the proposed model converts these judgments into structured q-CDNNIV information for aggregation and ranking. Hence, the proposed approach can be extended to other decision-making fields, including healthcare, energy planning, supplier selection, project evaluation, and engineering system assessment.

Step 2. According to Tables 7–10, the normalized decision values are as follows:

$$\overrightarrow{\vartheta_{kj}} = \frac{\vartheta_{kj}}{\max_k(\vartheta_{kj})}, \overrightarrow{\tau_{kj}} = \frac{\tau_{kj}}{\max_k(\tau_{kj})} \cdot \left(\frac{\tau_{kj}}{\vartheta_{kj}}\right), where\ k = 1,2,3,4,5\ and\ j = 1,2,3,4.$$

Suppose to find

$$\overrightarrow{\vartheta_{k1}} = \frac{\vartheta_{k1}}{\max_k(\vartheta_{k1})} = \frac{0.8}{\max\{0.8, 0.7, 0.85, 0.6, 0.9\}} = 0.8889$$

and

$$\overrightarrow{\tau_{k1}} = \frac{\tau_{k1}}{\max_k(\tau_{k1})} \cdot \left(\frac{\tau_{k1}}{\vartheta_{k1}}\right) = \left(\frac{0.75}{\max\{0.8, 0.7, 0.85, 0.6, 0.9\}}\right) \times \left(\frac{0.75}{0.8}\right) = 0.8272.$$

Similarly, we can find the other normalized values.

Step 3. Find the values of $\left(\bigoplus_{k=1}^{4} \wp_k \vartheta_k, \bigoplus_{k=1}^{4} \wp_k \tau_k\right)$ (q-CDNNIVWA). Now,

$$\begin{aligned}\bigoplus_{k=1}^{4} \wp_k \vartheta_k &= \wp_1\vartheta_1 \bigoplus \wp_2\vartheta_2 \bigoplus \wp_3\vartheta_3 \bigoplus \wp_4\vartheta_4 \\ &= (0.4 \times 0.8889) + (0.3 \times 0.8889) + (0.2 \times 0.8757) + (0.1 \times 0.7059) \\ &= 0.8678,\end{aligned}$$

and

$$\begin{aligned}\bigoplus_{k=1}^{4} \wp_k \tau_k &= \wp_1\tau_1 \bigoplus \wp_2\tau_2 \bigoplus \wp_3\tau_3 \bigoplus \wp_4\tau_4 \\ &= (0.4 \times 0.8272) + (0.3 \times 0.5402) + (0.2 \times 0.7347) + (0.1 \times 0.6302) \\ &= 0.7029.\end{aligned}$$

Hence, $\left(\bigoplus_{k=1}^{4} \wp_k \vartheta_k, \bigoplus_{k=1}^{4} \wp_k \tau_k\right) = (0.8678, 0.7029)$.

Similarly, we can find the other normalized values.

If $q = 1$, then $\begin{bmatrix} \sqrt{1 - \odot_{k=1}^{4}\left(1 - (\alpha R_k^{Tl})^2\right)^{\wp_k}}\, e^{i2\pi \sqrt[2]{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^2\right)^{\xi_k}}} \\ \sqrt{1 - \odot_{k=1}^{4}\left(1 - (\alpha R_k^{Tu})^2\right)^{\wp_k}}\, e^{i2\pi \sqrt[2]{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^2\right)^{\xi_k}}} \end{bmatrix}$.

Now, $\sqrt{1 - \odot_{k=1}^{4}\left(1 - (\alpha R_k^{Tl})^2\right)^{\wp_k}}\, e^{i2\pi \sqrt{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^2\right)^{\xi_k}}}$

$$\begin{aligned}&= \sqrt{1 - [(1-(0.495)^2)^{0.4} \times (1-(0.3825)^2)^{0.3} \times (1-(0.2)^2)^{0.2} \times (1-(0.575)^2)^{0.1}]} \\ &\quad e^{i2\pi\sqrt{1-[(1-(0.44)^2)^{0.4}\times(1-(0.315)^2)^{0.3}\times(1-(0.28)^2)^{0.2}\times(1-(0.1925)^2)^{0.1}]}} \\ &= 0.396 e^{i2\pi(0.3585)}.\end{aligned}$$

Now, $\sqrt{1 - \odot_{k=1}^{4}\left(1 - (\alpha R_k^{Tu})^2\right)^{\wp_k}}\, e^{i2\pi \sqrt{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^2\right)^{\xi_k}}}$

$$\begin{aligned}&= \sqrt{1 - [(1-(0.5225)^2)^{0.4} \times (1-(0.405)^2)^{0.3} \times (1-(0.26)^2)^{0.2} \times (1-(0.1925)^2)^{0.1}]} \\ &\quad e^{i2\pi\sqrt{1-[(1-(0.4675)^2)^{0.4}\times(1-(0.3375)^2)^{0.3}\times(1-(0.30)^2)^{0.2}\times(1-(0.2275)^2)^{0.1}]}} \\ &= \sqrt{1 - [(0.727)^{0.4} \times (0.8360)^{0.3} \times (0.9324)^{0.2} \times (0.9629)^{0.1}]}\, e^{i2\pi\sqrt{1-[(0.7814)^{0.4}\times(0.8861)^{0.3}\times(0.91)^{0.2}\times(0.9482)^{0.1}]}} \\ &= 0.4248 e^{i2\pi(0.3835)}.\end{aligned}$$

Table 7
Normalized decision values of 1st attribute.

	U_1
A	$\langle(0.8889, 0.8272); [0.495e^{i2\pi(0.44)}, 0.5225e^{i2\pi(0.4675)}], [0.39e^{i2\pi(0.39)}, 0.42e^{i2\pi(0.42)}], [0.325e^{i2\pi(0.325)}, 0.35e^{i2\pi(0.375)}]\rangle$
B	$\langle(0.7778, 0.5084); [0.32e^{i2\pi(0.34)}, 0.34e^{i2\pi(0.36)}], [0.35e^{i2\pi(0.35)}, 0.375e^{i2\pi(0.4)}], [0.3575e^{i2\pi(0.3575)}, 0.39e^{i2\pi(0.39)}]\rangle$
C	$\langle(0.9444, 0.8858); [0.35e^{i2\pi(0.375)}, 0.375e^{i2\pi(0.4)}], [0.2475e^{i2\pi(0.33)}, 0.275e^{i2\pi(0.3575)}], [0.42e^{i2\pi(0.42)}, 0.45e^{i2\pi(0.45)}]\rangle$
D	$\langle(0.6667, 0.3971); [0.51e^{i2\pi(0.48)}, 0.54e^{i2\pi(0.51)}], [0.27e^{i2\pi(0.2025)}, 0.2925e^{i2\pi(0.2475)}], [0.3575e^{i2\pi(0.44)}, 0.44e^{i2\pi(0.4675)}]\rangle$
E	$\langle(1, 0.9444); [0.4125e^{i2\pi(0.385)}, 0.495e^{i2\pi(0.4125)}], [0.25e^{i2\pi(0.325)}, 0.275e^{i2\pi(0.35)}], [0.24e^{i2\pi(0.45)}, 0.33e^{i2\pi(0.48)}]\rangle$

Table 8
Normalized decision values of 2nd attribute.

	U_2
A	$\langle(0.8889, 0.5402); [0.3825e^{i2\pi(0.315)}, 0.405e^{i2\pi(0.3375)}], [0.35e^{i2\pi(0.3)}, 0.425e^{i2\pi(0.375)}], [0.44e^{i2\pi(0.3025)}, 0.4675e^{i2\pi(0.3575)}]\rangle$
B	$\langle(0.9444, 0.4202); [0.455e^{i2\pi(0.455)}, 0.49e^{i2\pi(0.49)}], [0.27e^{i2\pi(0.2925)}, 0.36e^{i2\pi(0.36)}], [0.21e^{i2\pi(0.2275)}, 0.2275e^{i2\pi(0.2625)}]\rangle$
C	$\langle(1, 0.7778); [0.4875e^{i2\pi(0.39)}, 0.52e^{i2\pi(0.4225)}], [0.28e^{i2\pi(0.245)}, 0.315e^{i2\pi(0.2625)}], [0.165e^{i2\pi(0.225)}, 0.195e^{i2\pi(0.24)}]\rangle$
D	$\langle(0.7778, 0.7347); [0.56e^{i2\pi(0.385)}, 0.595e^{i2\pi(0.42)}], [0.3375e^{i2\pi(0.2925)}, 0.3825e^{i2\pi(0.315)}], [0.28e^{i2\pi(0.28)}, 0.32e^{i2\pi(0.34)}]\rangle$
E	$\langle(0.8889, 0.875); [0.51e^{i2\pi(0.24)}, 0.54e^{i2\pi(0.33)}], [0.3e^{i2\pi(0.18)}, 0.32e^{i2\pi(0.26)}], [0.4125e^{i2\pi(0.44)}, 0.4675e^{i2\pi(0.4675)}]\rangle$

Hence,

$$\left[\begin{array}{l}\sqrt{1-\odot_{k=1}^{4}\left(1-(\alpha R_k^{Tl})^2\right)^{\wp_k}}\, e^{i2\pi\sqrt{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^2\right)^{\xi_k}}} \\ \sqrt{1-\odot_{k=1}^{4}\left(1-(\alpha R_k^{Tu})^2\right)^{\wp_k}}\, e^{i2\pi\sqrt{1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^2\right)^{\xi_k}}}\end{array}\right] = [0.396e^{i2\pi(0.3585)}, 0.4248e^{i2\pi(0.3835)}].$$

Similarly,

$$\left[\begin{array}{l}\left(1-\odot_{k=1}^{4}\left(1-(\alpha R_k^{Il})\right)^{\wp_k}\right) e^{i2\pi\left(1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Il}}{2\pi}\right)\right)^{\xi_k}\right)} \\ \left(1-\odot_{k=1}^{4}\left(1-(\alpha R_k^{Iu})\right)^{\wp_k}\right) e^{i2\pi\left(1-\odot_{k=1}^{4}\left(1-\left(\frac{\alpha J_k^{Iu}}{2\pi}\right)\right)^{\xi_k}\right)}\end{array}\right] = [0.3982e^{i2\pi(0.3595)}, 0.4419e^{i2\pi(0.4144)}].$$

Similarly,

$$\left[\begin{array}{l}\odot_{k=1}^{4}(\gamma R_k^{Fl})^{\wp_k} e^{i2\pi\odot_{k=1}^{4}\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}}, \\ \odot_{k=1}^{4}(\gamma R_k^{Fu})^{\wp_k} e^{i2\pi\odot_{k=1}^{4}\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}}\end{array}\right] = [0.3331e^{i2\pi(0.3204)}, 0.3617e^{i2\pi(0.3745)}].$$

The q-CDNNIVWA operator is shown in Table 11.

Step 4. The q-CDNNIVWA $A = \Big[\Big\langle (0.8678, 0.7029); [0.396e^{i2\pi(0.3585)}, 0.4248e^{i2\pi(0.3835)}], [0.3982e^{i2\pi(0.3595)}, 0.4419e^{i2\pi(0.4144)}], [0.3331e^{i2\pi(0.3204)}, 0.3617e^{i2\pi(0.3745)}] \Big\rangle\Big]$.

The score function

$$\begin{aligned}\mathbb{S}_1 &= \frac{0.8678+0.7029}{4}\left[\begin{array}{l}\left(\frac{(0.3690)^2+(0.4248)^2}{2} - \frac{(0.3982)^2+(0.4419)^2}{2} + 2 - \frac{(0.3331)^2+(0.3617)^2}{2}\right) \\ +\left(\frac{(0.3585)^2+(0.3835)^2}{2} - \frac{(0.3595)^2+(0.4144)^2}{2} - \frac{(0.3204)^2+(0.3745)^2}{2}\right)\end{array}\right] \\ &= 0.6819.\end{aligned}$$

The score values are calculated as follows: $\mathbb{S}_1 = 0.6819$, $\mathbb{S}_2 = 0.6326$, $\mathbb{S}_3 = 0.8454$, $\mathbb{S}_4 = 0.7133$ and $\mathbb{S}_5 = 0.8471$. Alternatives and their corresponding ranks are listed below: $E \succ C \succ D \succ A \succ B$. Therefore, E is the best option. To more easily find the q-CDNNIVWA operator ranking results. The options are mentioned along with their ranks and scoring values. The q-CDNNIVWA operator yields ranking results for the alternatives that are essentially the same, the optimum option. The q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG operators can also be used.

7. Comparison analysis

In this section, we provide a detailed comparison of the proposed q-CDNNIVS-based MADM framework with existing decision-making methods. The goal is to determine the superiority and effectiveness of the proposed approach in handling complex, uncertain, and ambiguous data. The decision-making process often involves managing imprecise and contradictory data, and it is essential to use an approach that delivers consistent, reliable results even under such conditions. Our proposed framework integrates multiple types of information, truth, indeterminacy, falsity, interval-valued data, and normal fuzzy information within a unified framework, offering a significant advantage over traditional methods.

Table 9
Normalized decision values of 3rd attribute.

	U_3
A	$\langle(0.875, 0.7347), [0.2e^{i2\pi(0.28)}, 0.26e^{i2\pi(0.30)}], [0.39e^{i2\pi(0.33)}, 0.42e^{i2\pi(0.42)}], [0.225e^{i2\pi(0.325)}, 0.25e^{i2\pi(0.4)}]\rangle$
B	$\langle(0.6875, 0.526), [0.245e^{i2\pi(0.2625)}, 0.2625e^{i2\pi(0.28)}], [0.35e^{i2\pi(0.3)}, 0.375e^{i2\pi(0.4)}], [0.2e^{i2\pi(0.22)}, 0.24e^{i2\pi(0.26)}]\rangle$
C	$\langle(0.8125, 0.5495), [0.4e^{i2\pi(0.325)}, 0.425e^{i2\pi(0.35)}], [0.33e^{i2\pi(0.385)}, 0.3575e^{i2\pi(0.4675)}], [0.1225e^{i2\pi(0.1575)}, 0.14e^{i2\pi(0.245)}]\rangle$
D	$\langle(0.9375, 0.9333), [0.24e^{i2\pi(0.22)}, 0.26e^{i2\pi(0.3)}], [0.3575e^{i2\pi(0.4875)}, 0.4225e^{i2\pi(0.52)}], [0.3e^{i2\pi(0.25)}, 0.325e^{i2\pi(0.325)}]\rangle$
E	$\langle(1, 0.7545), [0.24e^{i2\pi(0.135)}, 0.255e^{i2\pi(0.195)}], [0.28e^{i2\pi(0.26)}, 0.34e^{i2\pi(0.28)}], [0.225e^{i2\pi(0.315)}, 0.2475e^{i2\pi(0.3825)}]\rangle$

Table 10
Normalized decision values of 4th attribute.

	U_4
A	$\langle(0.7059, 0.6302); [0.1575e^{i2\pi(0.1925)}, 0.1925e^{i2\pi(0.2275)}], [0.56e^{i2\pi(0.455)}, 0.595e^{i2\pi(0.49)}], [0.35e^{i2\pi(0.35)}, 0.4e^{i2\pi(0.375)}]\rangle$
B	$\langle(0.9412, 0.8789); [0.24e^{i2\pi(0.26)}, 0.26e^{i2\pi(0.3)}], [0.27e^{i2\pi(0.27)}, 0.315e^{i2\pi(0.2925)}], [0.21e^{i2\pi(0.2275)}, 0.2625e^{i2\pi(0.245)}]\rangle$
C	$\langle(0.8824, 0.8167); [0.25e^{i2\pi(0.3)}, 0.35e^{i2\pi(0.4)}], [0.22e^{i2\pi(0.28)}, 0.3e^{i2\pi(0.3)}], [0.26e^{i2\pi(0.2)}, 0.28e^{i2\pi(0.22)}]\rangle$
D	$\langle(1, 0.9412); [0.3575e^{i2\pi(0.44)}, 0.44e^{i2\pi(0.4675)}], [0.225e^{i2\pi(0.375)}, 0.325e^{i2\pi(0.4)}], [0.27e^{i2\pi(0.27)}, 0.36e^{i2\pi(0.2925)}]\rangle$
E	$\langle(0.9412, 0.8789); [0.315e^{i2\pi(0.3375)}, 0.3825e^{i2\pi(0.36)}], [0.4225e^{i2\pi(0.52)}, 0.455e^{i2\pi(0.5525)}], [0.2625e^{i2\pi(0.245)}, 0.2975e^{i2\pi(0.2625)}]\rangle$

Table 11
q-CDNNIVWA operator.

	q-CDNNIVWA operator (q = 1)
A	$\langle(0.8678, 0.7029); [0.396e^{i2\pi(0.3585)}, 0.4248e^{i2\pi(0.3835)}], [0.3982e^{i2\pi(0.3595)}, 0.4419e^{i2\pi(0.4144)}], [0.3331e^{i2\pi(0.3204)}, 0.3617e^{i2\pi(0.3745)}]\rangle$
B	$\langle(0.8261, 0.5225); [0.3505e^{i2\pi(0.3615)}, 0.3764e^{i2\pi(0.3885)}], [0.3191e^{i2\pi(0.3154)}, 0.3647e^{i2\pi(0.3781)}], [0.2573e^{i2\pi(0.2708)}, 0.2894e^{i2\pi(0.3048)}]\rangle$
C	$\langle(0.9285, 0.7792); [0.4007e^{i2\pi(0.3637)}, 0.4335e^{i2\pi(0.3978)}], [0.2718e^{i2\pi(0.3124)}, 0.3067e^{i2\pi(0.3495)}], [0.2364e^{i2\pi(0.2658)}, 0.2644e^{i2\pi(0.3072)}]\rangle$
D	$\langle(0.7875, 0.66); [0.4776e^{i2\pi(0.4103)}, 0.5124e^{i2\pi(0.4464)}], [0.3047e^{i2\pi(0.3127)}, 0.3509e^{i2\pi(0.3463)}], [0.3119e^{i2\pi(0.3268)}, 0.3689e^{i2\pi(0.377)}]\rangle$
E	$\langle(0.9608, 0.8791); [0.4133e^{i2\pi(0.3052)}, 0.4662e^{i2\pi(0.3502)}], [0.2901e^{i2\pi(0.2956)}, 0.3217e^{i2\pi(0.3355)}], [0.2812e^{i2\pi(0.3917)}, 0.3423e^{i2\pi(0.4284)}]\rangle$

Table 12
Comparison of our proposed methods with some existing methods.

Proposed method ($q = 1$)	q-CDNNIVWA	q-CDNNIVWG	q-GCDNNIVWA	q-GCDNNIVWG
Ranking	$E > C > D > A > B$	$E > C > D > A > B$	$E > C > D > A > B$	$E > C > D > A > B$
Palanikumar et al. (2022)	PNSNIVWA	PNSNIVWG	GPNSNIVWA	GPNSNIVWG
Ranking	$E > C > A > D > B$	$E > C > D > B > A$	$E > C > A > D > B$	$E > C > D > B > A$
Palanikumar et al. (2023)	SVNWA	SVNWG	GSVNWA	GSVNWG
Ranking	$E > C > A > D > B$	$E > C > D > B > A$	$E > C > A > D > B$	$E > C > D > B > A$
Yang and Chang (2020)	IVPNFWA	IVPNFWG	GIVPNFWA	GIVPNFWG
Ranking	$E > C > A > D > B$	$E > C > A > B > D$	$E > C > A > D > B$	$E > C > A > B > D$

We have introduced several AOs (q-CDNNIVWA, q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG) that enhance the robustness and flexibility of the decision-making processes. These operators are key to combining multiple sources of uncertain data, enabling decision-makers to select the optimal alternative under various conditions. In this comparison, we evaluate the performance of these operators alongside existing methods to determine how well they handle uncertainty and ambiguity.

Table 12 present comparative rankings for the proposed and existing methods. These results demonstrate the stability and discriminative power of the proposed method, highlighting its ability to yield consistent results across diverse decision-making scenarios.

- **Stability of the proposed method**
  Table 12 summarizes the ranking of the proposed methods using different AOs. The key observation from these rankings is that, across all four AOs, the score-based rankings consistently follow the same order of alternatives, indicating that the q-CDNNIVS-based framework produces reliable, consistent results regardless of the chosen AO. This consistency is vital in decision-making processes, where small changes in data or evaluation criteria should not result in drastic shifts in the ranking of alternatives. This result demonstrates the robustness of the framework, especially when evaluating multiple attributes under certain conditions.
  The existing methods yield different rankings for the alternatives, demonstrating that their decision-making frameworks are less stable and less consistent than the proposed approach. While these methods still generally identify the best and worst alternatives, the variability in the intermediate alternatives suggests they may struggle to handle complex, inconsistent data as effectively as the proposed framework.
- **Discriminative power and flexibility**
  The proposed q-CDNNIVS-based decision-making framework demonstrates strong discriminative power and flexibility in handling complex MADM problems. It effectively distinguishes between intermediate alternatives, even when their differences are subtle. This capability is particularly important in real-world decision environments, where alternatives often exhibit close or overlapping performance across multiple criteria.
  The framework shows consistent ranking behavior when using only the score function. It reliably identifies the best alternative as the optimal choice and the least favorable alternative as the worst. This stable ranking structure ensures robust decision outcomes under uncertainty, vagueness, and incomplete information.

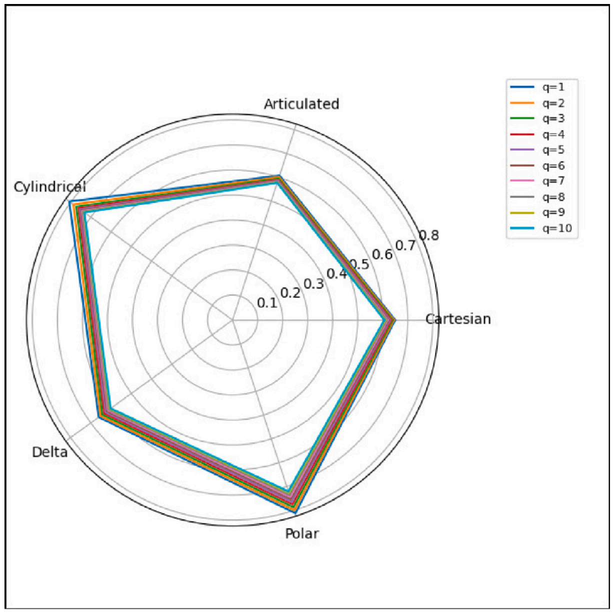

Fig. 4. Relative closeness values for WG operator.

Unlike many existing methods, which produce fluctuating rankings across different evaluation settings, the proposed approach maintains stable rankings across all tested cases. This consistency highlights its reliability and practical suitability for MADM problems involving ambiguous and high-dimensional information.

- **Summary of ranking performance**
  The results from Table 12 clearly demonstrate the effectiveness of the proposed method in producing stable, consistent ranking across score functions. The key findings can be summarized as follows:

1. The proposed method consistently yields the same ranking order across all four AOs, showing its stability and reliability in handling complex decision-making problems.
2. The existing methods exhibit variability in their rankings across AOs, suggesting they are less stable and less reliable than the proposed approach.
3. The proposed method demonstrates superior power, particularly in distinguishing between intermediate alternatives, making it a more effective tool for decision-makers facing uncertain and inconsistent data.

The comparison analysis confirms that the q-CDNNIVS-based MADM method is both stable and reliable, providing consistent rankings across all AOs. Its ability to maintain stability and accuracy under uncertain conditions gives it a distinct advantage over existing methods, which show variability in their results. The proposed framework offers a more discriminative and flexible decision-making tool, particularly valuable in complex decision environments where multiple conflicting criteria must be considered.

By maintaining consistent results and providing clear distinctions between alternatives, the proposed method is a robust decision-support tool for applications in industrial robot selection and other decision-making scenarios. The performance of the proposed approach, especially in handling complex, uncertain, and ambiguous data, underscores its potential as a practical and effective solution for real-world decision-making problems.

8. Sensitivity analysis

8.1. Sensitivity analysis of the parameter q

The parameter q controls the admissible information space of the proposed q-CDNNIV framework. Therefore, sensitivity analysis with respect to q is necessary to examine how the decision results respond to different uncertainty-control settings. A smaller value of q imposes a stricter constraint on the truth, indeterminacy, and falsity degrees, while a larger value of q provides a wider admissible space for uncertain information. Thus, the parameter q reflects the decision-maker's attitude toward uncertainty rather than acting as an arbitrary numerical value.

Table 13 presents the score values and ranking orders obtained by q-CDNNIVWG operators for different values of q. The results show that the ranking pattern changes gradually as q increases. When $1 \leq q \leq 4$, the ranking order is $E \succ C \succ D \succ A \succ B$. When $5 \leq q \leq 7$, the ranking becomes $E \succ C \succ A \succ D \succ B$. This indicates that alternative E remains the best option under stricter uncertainty-control settings, while A and D exchange their position in the middle ranks. For $8 \leq q \leq 14$, the best alternative changes from E to C, and the ranking order becomes $C \succ E \succ A \succ D \succ B$. This shift indicates that alternative C performs better when the admissible space for uncertainty becomes wider. For larger values of q, similar gradual variations in the ranking order are observed, although the changes mainly occur among the middle-ranked alternatives, while alternatives C and E consistently remain the dominant candidates across different parameter settings.

Figs. 4, 5, and 6 further illustrate the variation in score values under the q-CDNNIVWG operator for different q-values. These figures supported the results reported in Table 13 and show how the alternatives respond to changes in the uncertainty-control parameter.

Table 13
Score functions and orders of ranking by q-CDNNIVWG operators with different q values.

q-values	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{S}_4$	$\mathbb{S}_5$	Order
$q = 1$	0.6495	0.6068	0.8066	0.6608	0.8122	$E > C > D > A > B$
$q = 2$	0.6443	0.6005	0.7867	0.6522	0.7996	$E > C > D > A > B$
$q = 3$	0.6390	0.5947	0.7717	0.6431	0.7868	$E > C > D > A > B$
$q = 4$	0.6339	0.5901	0.7610	0.6346	0.7748	$E > C > D > A > B$
$q = 5$	0.6289	0.5865	0.7532	0.6270	0.7636	$E > C > A > D > B$
$q = 6$	0.6241	0.5838	0.7471	0.6204	0.7534	$E > C > A > D > B$
$q = 7$	0.6195	0.5816	0.7422	0.6147	0.7440	$E > C > A > D > B$
$q = 8$	0.6150	0.5798	0.7381	0.6098	0.7354	$C > E > A > D > B$
$q = 9$	0.6106	0.5784	0.7345	0.6057	0.7277	$C > E > A > D > B$
$q = 10$	0.6065	0.5771	0.7315	0.6020	0.7208	$C > E > A > D > B$
$q = 11$	0.6025	0.5761	0.7288	0.5989	0.7146	$C > E > A > D > B$
$q = 12$	0.5988	0.5752	0.7263	0.5962	0.7090	$C > E > A > D > B$
$q = 13$	0.5953	0.5744	0.7242	0.5937	0.7040	$C > E > A > D > B$
$q = 14$	0.5921	0.5736	0.7222	0.5916	0.5996	$C > E > A > D > B$
$q = 15$	0.5891	0.5730	0.7205	0.5897	0.6956	$C > E > D > A > B$
$q = 16$	0.5865	0.5725	0.7189	0.5880	0.6919	$C > E > D > A > B$
$q = 17$	0.6050	0.5723	0.7174	0.5867	0.6887	$C > E > A > D > B$
$q = 18$	0.6026	0.6118	0.7161	0.6063	0.6857	$C > E > B > D > A$
$q = 19$	0.6012	0.6116	0.7149	0.6051	0.6829	$C > E > B > D > A$
$q = 20$	0.6273	0.6596	0.7138	0.6039	0.6808	$C > E > B > A > D$
$q = 21$	0.6254	0.6594	0.7841	0.6028	0.7151	$C > E > B > A > D$
$q = 22$	0.6228	0.6592	0.7833	0.6011	0.7128	$C > E > B > A > D$
$q = 23$	0.6577	0.6591	0.8650	0.6663	0.7561	$C > E > D > B > A$
$q = 24$	0.6970	0.6589	0.8644	0.7021	0.8027	$C > E > D > A > B$
$q = 25$	0.6958	0.6588	0.8639	0.7015	0.8524	$C > E > D > A > B$
$q = 30$	0.6905	0.6583	0.8618	0.6991	0.8467	$C > E > D > A > B$
$q = 35$	0.6865	0.6971	0.9055	0.7179	0.8425	$C > E > D > B > A$
$q = 40$	0.6834	0.7457	0.9338	0.7166	0.8392	$C > E > B > D > A$
$q = 45$	0.7174	0.7457	0.9338	0.7447	0.9166	$C > E > B > D > A$
$q = 50$	0.7584	0.7457	0.9765	0.7830	0.9152	$C > E > D > A > B$
$q = 55$	0.7573	0.7457	0.9765	0.8272	0.9714	$C > E > D > A > B$

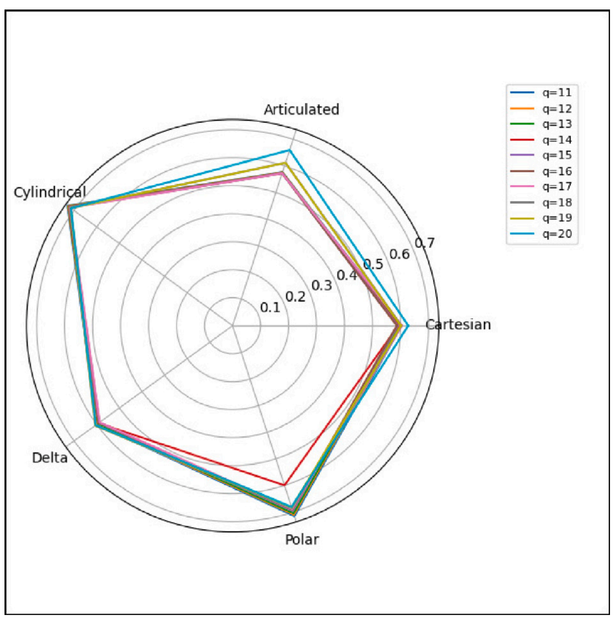

Fig. 5. Relative closeness values for WG operator.

Overall, the sensitivity analysis shows that alternatives C and E are the most dominant candidates across different q-settings. Under smaller q-values, alternative E obtains the highest rank. However, when q increases, alternative C becomes the most preferred option in most cases. This result confirms that the proposed model is sensitive to the decision-maker's uncertainty while maintaining a stable decision structure, as the top two alternatives remain C and E across all tested cases. Therefore, the final decision should be interpreted together with the selected q-value, since q directly affects the admissible uncertainty space and may influence the ranking of alternatives.

Table 14 demonstrates that the ranking results obtained from the q-CDNNIVWA operator vary with changes in the parameter q. For $1 \leq q \leq 5$, the ranking remains stable as $E \succ C \succ D \succ A \succ B$. This indicates that alternative E consistently performs best under lower values of q, where the model imposes a stricter structure on the representation of uncertainty. When q increases to the interval $6 \leq q \leq 20$, a slight shift in ranking is observed, and the order becomes $C \succ E \succ D \succ A \succ B$. This transition shows that alternative C starts to dominate as the admissible uncertainty space expands, while the relative positions of the remaining alternatives remain largely stable. while the relative positions of the remaining alternatives remain

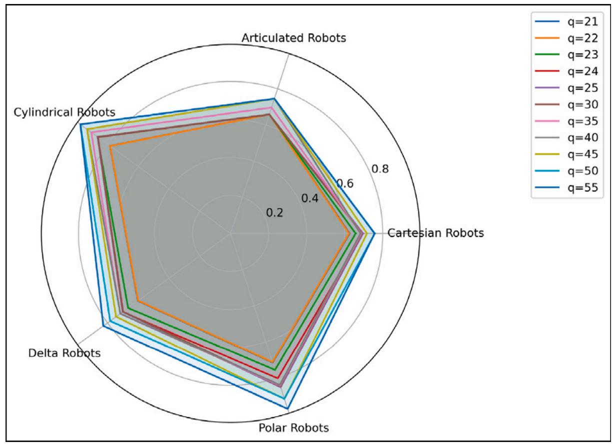

Fig. 6. Relative closeness values for WG operator.

Table 14
Score functions and orders of ranking by q-CDNNIVWA operators with different q values.

q-values	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{S}_4$	$\mathbb{S}_5$	Order
$q = 1$	0.6819	0.6326	0.8454	0.7133	0.8471	$E > C > D > A > B$
$q = 2$	0.6922	0.6401	0.8478	0.7188	0.8538	$E > C > D > A > B$
$q = 3$	0.6997	0.6474	0.8500	0.7212	0.8568	$E > C > D > A > B$
$q = 4$	0.7044	0.6534	0.8518	0.7216	0.8569	$E > C > D > A > B$
$q = 5$	0.7072	0.6582	0.8530	0.7208	0.8549	$E > C > D > A > B$
$q = 6$	0.7084	0.6618	0.8538	0.7196	0.8517	$C > E > D > A > B$
$q = 7$	0.7086	0.6647	0.8543	0.7183	0.8479	$C > E > D > A > B$
$q = 8$	0.7080	0.6668	0.8544	0.7170	0.8438	$C > E > D > A > B$
$q = 9$	0.7069	0.6685	0.8544	0.7158	0.8398	$C > E > D > A > B$
$q = 10$	0.7054	0.6699	0.8542	0.7147	0.8359	$C > E > D > A > B$
$q = 15$	0.6967	0.6738	0.8525	0.7107	0.8207	$C > E > D > A > B$
$q = 20$	0.6892	0.6756	0.8194	0.7084	0.7792	$C > E > D > A > B$
$q = 30$	0.5022	0.5324	0.6761	0.6196	0.6077	$C > D > E > B > A$
$q = 40$	0.4951	0.6210	0.7487	0.5218	0.6002	$C > B > E > D > A$
$q = 50$	0.5706	0.6210	0.7918	0.5907	0.6764	$C > E > B > D > A$
$q = 60$	0.5681	0.6210	0.7973	0.6165	0.7957	$C > E > B > D > A$

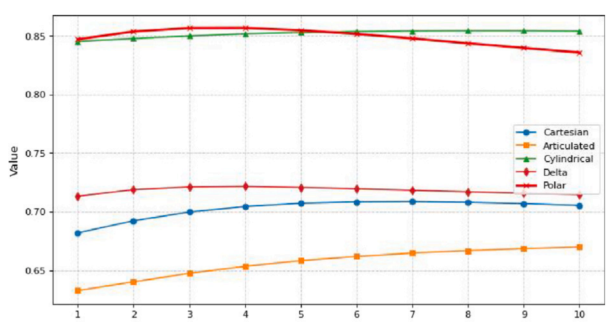

Fig. 7. Relative closeness values for WA operator.

largely stable. For higher values of q (i.e., $q \geq 30$), more pronounced variations in ranking appear due to increased flexibility in the aggregation process. However, the top-performing alternatives remain C and E, indicating strong robustness of these two options across different uncertainty settings. The remaining alternatives show only minor positional changes, which do not affect the overall decision stability.

Figs. 7 and 8 further illustrate the behavior of proximity values under the q-CDNNIVWA operator. These figures confirm that changes in q influence the magnitude of the aggregated scores, particularly when the differences among alternatives are small. However, the overall ranking structure remains consistent across most parameter settings, indicating that the proposed method maintains a strong level of stability.

These results show that changes in q can affect the ranking order, especially when the scores of alternatives are close. This does not indicate poor method stability. Instead, it shows that the proposed model can capture different decision attributes under different uncertainty levels. Therefore, the role of q is to provide controllable flexibility in the decision-making process.

To interpret the sensitivity results, we consider two aspects: local stability and global sensitivity. If the best alternative remains unchanged within a practical range of q, the results are locally stable. If ranking changes occur only under larger changes in q, this indicates that the method is sensitive

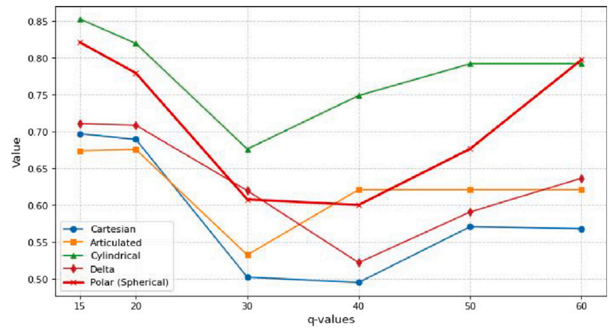

Fig. 8. Relative closeness values for WA operator.

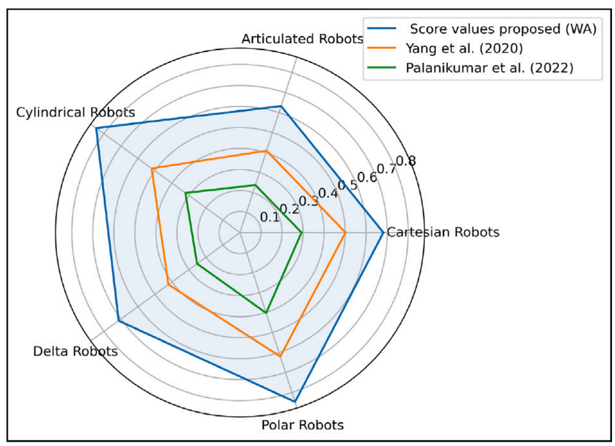

Fig. 9. Comparison of models based on score values of averaging operators.

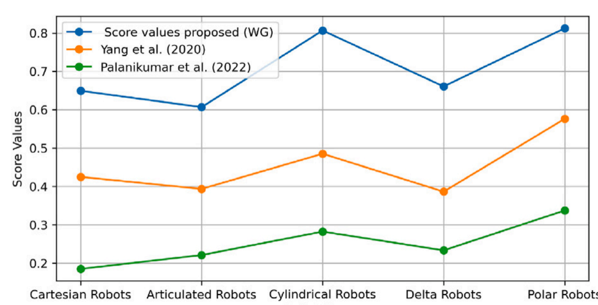

Fig. 10. Comparison of models based on score values of geometric operators.

to the selected uncertainty-control setting. In such cases, the decision-maker should select q based on the uncertainty level, risk preferences, and the practical requirements of the application.

Overall, the sensitivity analysis of the q-CDNNIVW operator demonstrates that alternative C and E remain the most competitive choices across all tested scenarios. This confirms the robustness of the proposed framework for handling complex, uncertain decision environments.

Figs. 9 and 10 further illustrate the influence of parameter variation on the scores obtained from both the proposed and existing models across different aggregation frameworks. These figures provide a clear visual representation of how the alternatives behave across varying values of q, allowing identification of stable ranking regions and transition points where ranking changes occur.

From these graphical results, it is observed that the sensitivity analysis offers additional decision support by evaluating the robustness of ranking outcomes across different uncertainty-control settings. In particular, the variation in ranking patterns indicates that the parameter q directly regulates the strictness of the uncertainty representation space. A lower value of q imposes a more constrained environment, while a higher value allows greater flexibility in handling uncertainty, potentially altering the comparative dominance of alternatives.

The results presented in Figs. 9 and 10 correspond to the scores obtained using the WA and WG operators, respectively. In both cases, the proposed model demonstrates consistent performance compared with existing approaches, particularly in identifying dominant alternatives across different parameter settings. The graphical comparison confirms that although minor fluctuations occur in intermediate values, the overall ranking structure remains stable within meaningful intervals of q.

Table 15
Various weight sets of criteria.

Weight sets	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$	Weight sets	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
W_1	0.4	0.3	0.2	0.1	W_{13}	0.4	0.2	0.1	0.3
W_2	0.4	0.2	0.3	0.1	W_{14}	0.3	0.4	0.1	0.2
W_3	0.4	0.1	0.3	0.2	W_{15}	0.3	0.4	0.2	0.1
W_4	0.4	0.1	0.2	0.3	W_{16}	0.3	0.2	0.1	0.4
W_5	0.3	0.2	0.4	0.1	W_{17}	0.2	0.4	0.1	0.3
W_6	0.3	0.1	0.2	0.4	W_{18}	0.2	0.3	0.1	0.4
W_7	0.3	0.1	0.4	0.2	W_{19}	0.1	0.4	0.3	0.2
W_8	0.2	0.4	0.3	0.1	W_{20}	0.1	0.4	0.2	0.3
W_9	0.2	0.3	0.4	0.1	W_{21}	0.1	0.2	0.4	0.3
W_{10}	0.2	0.1	0.3	0.4	W_{22}	0.1	0.4	0.2	0.3
W_{11}	0.2	0.1	0.4	0.3	W_{23}	0.1	0.3	0.4	0.2
W_{12}	0.4	0.3	0.1	0.2	W_{24}	0.1	0.3	0.2	0.4

Table 16
Different weighted values.

Weights	$\mathbb{S}_1$	$\mathbb{S}_2$	$\mathbb{S}_3$	$\mathbb{S}_4$	$\mathbb{S}_5$
W_1	0.6495	0.6068	0.8066	0.6608	0.8122
W_2	0.6550	0.5900	0.7752	0.6526	0.8055
W_3	0.6312	0.6013	0.7624	0.6660	0.8015
W_4	0.6021	0.6289	0.7805	0.6874	0.8029
W_5	0.6429	0.5892	0.7612	0.6717	0.7898
W_6	0.5651	0.6576	0.7844	0.7291	0.7867
W_7	0.6206	0.6010	0.7485	0.6858	0.7864
W_8	0.6195	0.6213	0.8090	0.7063	0.7836
W_9	0.6255	0.6041	0.7778	0.6991	0.779
W_{10}	0.5571	0.6576	0.7699	0.7507	0.7709
W_{11}	0.5837	0.6287	0.7522	0.7288	0.7718
W_{12}	0.6194	0.6342	0.8250	0.6831	0.8158
W_{13}	0.5959	0.6450	0.8118	0.6963	0.8104
W_{14}	0.6022	0.6511	0.8421	0.7102	0.8045
W_{15}	0.6313	0.6232	0.8238	0.6871	0.8011
W_{16}	0.5585	0.6735	0.8155	0.7380	0.7935
W_{17}	0.5641	0.6783	0.8453	0.7528	0.7878
W_{18}	0.5437	0.6899	0.8319	0.7674	0.7822
W_{19}	0.5816	0.6475	0.8121	0.7502	0.7687
W_{20}	0.5520	0.6766	0.8300	0.7736	0.7696
W_{21}	0.5685	0.6432	0.7681	0.7588	0.761
W_{22}	0.5552	0.6766	0.8300	0.7736	0.7696
W_{23}	0.5881	0.6305	0.7810	0.7431	0.7648
W_{24}	0.5359	0.6889	0.8167	0.7892	0.7646

Therefore, the sensitivity analysis not only assesses numerical variations but also provides interpretability into the stability of the decision-making framework. It clearly shows how robust the ranking is under different uncertainty-control configurations. The observed variations confirm that the parameter q plays a crucial role in controlling the admissible uncertainty space and directly influences aggregation behavior.

Hence, the q-based sensitivity analysis helps identify stable ranking intervals and supports decision-makers in selecting an appropriate q value based on the level of uncertainty in practical applications. This improves the interpretability, reliability, and flexibility of the proposed MADM framework.

8.2. Sensitivity analysis of criteria weights

In addition to the sensitivity analysis of parameter q, we also examine the effect of the criteria weights on the ranking results. Criteria weights represent the relative importance of each attribute in the decision-making process. Since different decision-makers may assign varying levels of importance to reach, payload, flexibility, speed, and standardization, a weight-based sensitivity analysis is necessary to assess the robustness of the proposed method.

Table 15 presents 24 different weight combinations, with weights $\wp_1 = 0.40$, $\wp_2 = 0.30$, $\wp_3 = 0.20$ and $\wp_4 = 0.10$. These combinations are used to test whether the ranking results remain consistent when the relative importance of the criteria changes. The corresponding ranking results are reported in Tables 16 and 17. Similarity values for different sets of criteria weights for WG operators in Fig. 11.

The results show that alternative E and C consistently appear as the leading candidates across most weight settings. This indicates that the proposed method identifies a stable group of dominant alternatives, even when the criteria weights vary. Although the first-ranked alternative may change between E and C, the lower-ranked alternatives remain relatively consistent in most cases. This suggests that the method is sensitive enough to reflect changes in decision preferences while still maintaining meaningful ranking consistency.

Therefore, the weight-sensitivity analysis confirms that the proposed method is not tied to a single weight scenario. Instead, it allows decision-makers to understand how changes in the importance of criteria affect the final ranking. This improves the transparency and practical usefulness of the proposed decision-making framework.

Table 17
Ranking of alternatives for various weight groupings.

	Ranked list		Ranked list		Ranked list
W_1	$E \succ C \succ D \succ B \succ A$	W_9	$E \succ C \succ D \succ A \succ B$	W_{17}	$C \succ E \succ D \succ B \succ A$
W_2	$E \succ C \succ A \succ D \succ B$	W_{10}	$E \succ C \succ D \succ B \succ A$	W_{18}	$C \succ E \succ D \succ B \succ A$
W_3	$E \succ C \succ D \succ A \succ B$	W_{11}	$E \succ C \succ D \succ B \succ A$	W_{19}	$C \succ E \succ D \succ B \succ A$
W_4	$E \succ C \succ D \succ B \succ A$	W_{12}	$C \succ E \succ D \succ B \succ A$	W_{20}	$C \succ E \succ D \succ B \succ A$
W_5	$E \succ C \succ D \succ A \succ B$	W_{13}	$C \succ E \succ D \succ B \succ A$	W_{21}	$C \succ E \succ D \succ B \succ A$
W_6	$E \succ C \succ D \succ B \succ A$	W_{14}	$C \succ E \succ D \succ B \succ A$	W_{22}	$C \succ E \succ D \succ B \succ A$
W_7	$E \succ C \succ D \succ A \succ B$	W_{15}	$C \succ E \succ D \succ A \succ B$	W_{23}	$C \succ E \succ D \succ B \succ A$
W_8	$C \succ E \succ D \succ B \succ A$	W_{16}	$C \succ E \succ D \succ B \succ A$	W_{24}	$C \succ D \succ E \succ B \succ A$

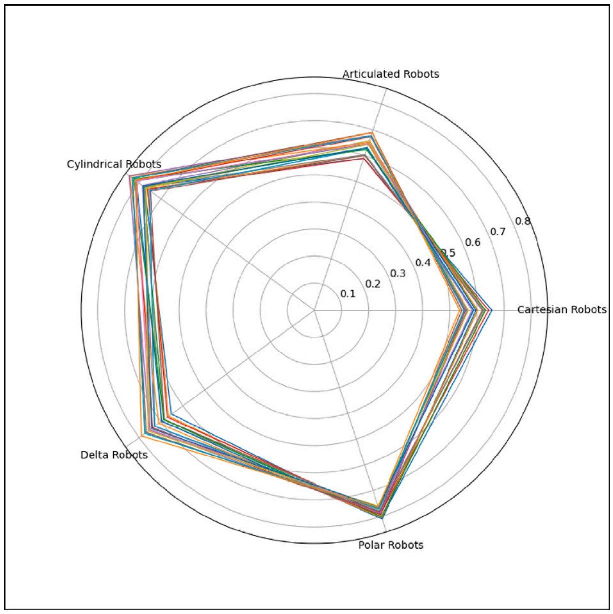

Fig. 11. Different sets of criteria weights for WG operators.

9. Advantages of the study

The proposed method offers several key advantages, particularly in minimizing information loss while analyzing significantly larger datasets. By incorporating reference parameters, these methods provide a more comprehensive evaluation of alternatives. Additionally, the use of distance and similarity assessments, considering both optimistic and pessimistic scenarios, with or without weighting components, enhances decision-making, enabling DMs to more accurately identify the best course of action. In other words, the DM is presented with a diverse range of options that reflect both positive and negative perspectives, supported by innovative measures of similarity and distance.

The q-CDNNIVWA, q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG are introduced to address these needs. A unique aspect of the proposed framework is its application to decision-making in robotics, which is modeled as a fuzzy set of steps and processes. This approach enables the consideration of various possibilities and alternatives, allowing DMs to compare available options based on a comprehensive set of attributes. The generalized weighted operator, central to this method, is based on a weighted aggregation model, offering flexibility for handling diverse scenarios.

This approach draws inspiration from complex decision challenges, expanding its scope to accommodate a broader range of circumstances. The weighted operators, which factor in both accuracy and score values, enable comparisons of ideal preferences against selected alternatives. By offering a more thorough and accurate decision-making process, the proposed method stands out from existing alternatives.

Furthermore, the applicability of this method extends beyond decision-making in industrial contexts, enabling the development of intelligent systems for image recognition and other practical applications. To validate the effectiveness of the proposed method, a comparison analysis is conducted. This comparison evaluates the method from two distinct perspectives, focusing on the robustness of the Aos that underpin our MADM method. This comparative evaluation highlights the strength and reliability of the proposed approach in handling complex decision scenarios.

9.1. Limitations of the study

Some limitations should be noted, despite the proposed q-CDNNIVS-based MADM architecture offering a versatile and trustworthy method for managing uncertainty in industrial robot selection.

1. The framework relies heavily on expert evaluations and criteria weights, which can introduce bias or inconsistency. Any inaccuracies or subjectivity in the expert judgments could affect the reliability of the decision-making process.

2. The choice of parameter q and the distribution of weights among criteria affect the final results. Careful adjustment across contexts is necessary, as a poor choice of these parameters can yield skewed or unreliable outcomes.

3. Although the framework is used in a case study on industrial robot selection, its effectiveness has not been thoroughly examined with larger datasets or across a variety of decision-making domains. To generalize the method, more thorough validation is required.

4. The model's incorporation of complex-valued and interval-valued neutrosophic data, along with sophisticated AOs, may obfuscate the computation and interpretation of outcomes. Its use in large-scale or less technically complex applications may be limited by this complexity.

10. Conclusions

In this work, we presented a q-CDNNIVSs for resolving DM problems in a complex neutrosophic environment. By using language concepts expressed as neutrosophic numbers to rate options against specific criteria, the suggested strategy effectively manages uncertainty. We created other AOs, such as q-CDNNIVWA, q-CDNNIVWG, q-GCDNNIVWA, and q-GCDNNIVWG, in addition to the unique q-CDNNIVS. These operators' robustness and application are guaranteed by fundamental characteristics, including commutativity, idempotency, boundedness, associativity, and monotonicity.

The findings in Table 12 illustrate the method's improved performance compared to existing methodologies, demonstrating the efficacy of the proposed strategy for solving MADM problems. The q-CDNNIVS model's addition of score and accuracy functions has demonstrated encouraging outcomes in solving practical DM problems. These functions offer a thorough and efficient way to assess options in DM situations, especially when combined with the new weighted operators. The ability to modify the parameters q and an increases the method's flexibility and makes it suitable for a range of DM scenarios. This study presents a numerical example that further validates the methodology's capacity to address difficult MADM problems, demonstrating its practical applicability and usefulness.

Future studies will focus on expanding the proposed approach to address a broader range of practical problems across fields such as manufacturing, business, and healthcare. To improve the adaptability and durability of our DM framework, we intend to investigate a range of fuzzy environments, including linguistic NSs, interval type-2 FSs, T-SFSs, complex soft sets, and hesitant fuzzy sets with extension. These extensions will ultimately help decision-makers in a variety of industries by expanding the applicability of our method to more complicated, real-world situations.

CRediT authorship contribution statement

Murugan Palanikumar: Validation, Software, Methodology, Investigation, Conceptualization. **Nasreen Kausar:** Writing – review & editing, Writing – original draft, Visualization, Project administration, Formal analysis, Data curation. **Tapan Senapati:** Writing – review & editing, Visualization, Validation, Supervision, Resources, Methodology, Conceptualization.

Ethical approval

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Theorem 4.1

If $m = 2$, then q-CDNNIVWA$(\widetilde{\Phi}_1, \widetilde{\Phi}_2) = \wp_1\widetilde{\Phi}_1 \bigoplus \wp_2\widetilde{\Phi}_2$, where

$$\wp_1\widetilde{\Phi}_1 = \left[\begin{array}{c} \left(\wp_1\vartheta_1, \wp_1\tau_1\right); \\ \left[\sqrt[2q]{1-\left(1-(\alpha R_1^{Tl})^{2q}\right)^{\wp_1}}\, e^{i2\pi \sqrt[2q]{1-\left(1-\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_1}}}, \sqrt[2q]{1-\left(1-(\alpha R_1^{Tu})^{2q}\right)^{\wp_1}}\, e^{i2\pi \sqrt[2q]{1-\left(1-\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_1}}}\right]; \\ \left[\sqrt[q]{1-\left(1-(\beta R_1^{Il})^{q}\right)^{\wp_1}}\, e^{i2\pi \sqrt[q]{1-\left(1-\left(\frac{\beta J_1^{Il}}{2\pi}\right)^{q}\right)^{\xi_1}}}, \sqrt[q]{1-\left(1-(\beta R_1^{Iu})^{q}\right)^{\wp_1}}\, e^{i2\pi \sqrt[q]{1-\left(1-\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^{q}\right)^{\xi_1}}}\right]; \\ \left[(\gamma R_1^{Fl})^{\wp_1} e^{i2\pi\left(\frac{\gamma J_1^{Fl}}{2\pi}\right)^{\xi_1}}, (\gamma R_1^{Fu})^{\wp_1} e^{i2\pi(\gamma R_1^{Fu})^{\wp_1}}\right] \end{array}\right]$$

$$\wp_2\widetilde{\Phi}_2=\left[\begin{array}{c}\left(\wp_2\vartheta_2,\wp_2\tau_2\right);\\ \left[\sqrt[2q]{1-\left(1-(\alpha R_2^{Tl})^{2q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[2q]{1-\left(1-\left(\frac{\alpha J_2^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_2}}},\sqrt[2q]{1-\left(1-(\alpha R_2^{Tu})^{2q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[2q]{1-\left(1-\left(\frac{\alpha J_2^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_2}}}\right];\\ \left[\sqrt[q]{1-\left(1-(\beta R_2^{Il})^{q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[q]{1-\left(1-\left(\frac{\beta J_2^{Il}}{2\pi}\right)^{q}\right)^{\xi_2}}},\sqrt[q]{1-\left(1-(\beta R_2^{Iu})^{q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[q]{1-\left(1-\left(\frac{\beta J_2^{Iu}}{2\pi}\right)^{q}\right)^{\xi_2}}}\right];\\ \left[(\gamma R_2^{Fl})^{\wp_2}e^{i2\pi\left(\frac{\gamma J_2^{Fl}}{2\pi}\right)^{\xi_2}},(\gamma R_2^{Fu})^{\wp_2}e^{i2\pi\left(\frac{\gamma J_2^{Fu}}{2\pi}\right)^{\xi_2}}\right]\end{array}\right].$$

Now, $\wp_1\widetilde{\Phi}_1\bigoplus\wp_2\widetilde{\Phi}_2=$

$$\left[\begin{array}{c}\left(\wp_1\vartheta_1+\wp_2\vartheta_2,\wp_1\tau_1+\wp_2\tau_2\right);\\ \left[\begin{array}{c}\sqrt[2q]{1-\left(1-(\alpha R_1^{Tl})^{2q}\right)^{\wp_1}\left(1-(\alpha R_2^{Tl})^{2q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[2q]{1-\left(1-\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_1}\left(1-\left(\frac{\alpha J_2^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_2}}}\\ \sqrt[2q]{1-\left(1-(\alpha R_1^{Tu})^{2q}\right)^{\wp_1}\left(1-(\alpha R_2^{Tu})^{2q}\right)^{\wp_2}}\,e^{i2\pi 1-\left(1-\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_1}\left(1-\left(\frac{\alpha J_2^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_2}}\end{array}\right],\\ \left[\begin{array}{c}\sqrt[q]{1-\left(1-(\beta R_1^{Il})^{q}\right)^{\wp_1}\left(1-(\beta R_2^{Il})^{q}\right)^{\wp_2}}\,e^{i2\pi\sqrt[q]{1-\left(1-\left(\frac{\beta J_1^{Il}}{2\pi}\right)^{q}\right)^{\xi_1}\left(1-\left(\frac{\beta J_2^{Il}}{2\pi}\right)^{q}\right)^{\xi_2}}}\\ \sqrt[q]{1-\left(1-(\beta R_1^{Iu})^{q}\right)^{\wp_1}\left(1-(\beta R_2^{Iu})^{q}\right)^{\wp_2}}\,e^{i2\pi 1-\left(1-\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^{q}\right)^{\xi_1}\left(1-\left(\frac{\beta J_2^{Iu}}{2\pi}\right)^{q}\right)^{\xi_2}}\end{array}\right],\\ \left[(\gamma R_1^{Fl})^{\wp_1}e^{i2\pi\left(\frac{\gamma J_1^{Fl}}{2\pi}\right)^{\xi_1}}\cdot(\gamma R_2^{Fl})^{\wp_2}e^{i2\pi\left(\frac{\gamma J_2^{Fl}}{2\pi}\right)^{\xi_2}},(\gamma R_1^{Fu})^{\wp_1}e^{i2\pi\left(\frac{\gamma J_1^{Fu}}{2\pi}\right)^{\xi_1}}\cdot(\gamma R_2^{Fu})^{\wp_2}e^{i2\pi\left(\frac{\gamma J_2^{Fu}}{2\pi}\right)^{\xi_2}}\right]\end{array}\right]$$

q-$CDNNIVWA(\widetilde{\Phi}_1,\widetilde{\Phi}_2)=$

$$\left[\begin{array}{c}\left(\bigoplus_{k=1}^{2}\wp_k\vartheta_k,\bigoplus_{k=1}^{2}\wp_k\tau_k\right);\\ \left[\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}}},\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}}}\right]\\ \left[\sqrt[q]{1-\odot_{k=1}^{2}\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}}},\sqrt[q]{1-\odot_{k=1}^{2}\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}}}\right]\\ \left[\odot_{k=1}^{2}(\gamma R_k^{Fl})^{\wp_k}e^{i2\pi\odot_{k=1}^{2}\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}},\odot_{k=1}^{2}(\gamma R_k^{Fu})^{\wp_k}e^{i2\pi\odot_{k=1}^{2}\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}}\right]\end{array}\right].$$

It is valid for $m\geq 3$, q-CDNNIVWA$(\widetilde{\Phi}_1,\widetilde{\Phi}_2,\ldots,\widetilde{\Phi}_p)=$

$$\left[\begin{array}{c}\left(\bigoplus_{k=1}^{p}\wp_k\vartheta_k,\bigoplus_{k=1}^{p}\wp_k\tau_k\right);\\ \left[\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}}},\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}}}\right],\\ \left[\sqrt[q]{1-\odot_{k=1}^{p}\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}}},\sqrt[q]{1-\odot_{k=1}^{p}\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}}\,e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}}}\right],\\ \left[\odot_{k=1}^{p}(\gamma R_k^{Fl})^{\wp_k}e^{i2\pi\odot_{k=1}^{p}\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}},\odot_{k=1}^{p}(\gamma R_k^{Fu})^{\wp_k}e^{i2\pi\odot_{k=1}^{p}\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}}\right]\end{array}\right].$$

If $m = p+1$, then q-CDNNIVWA $(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_p, \widetilde{\Phi}_{p+1})$

$$
\left[\begin{array}{c}
\left(\bigoplus_{k=1}^{p}\wp_k\vartheta_k+\wp_{p+1}\vartheta_{p+1}, \bigoplus_{k=1}^{p}\wp_k\tau_k+\wp_{p+1}\tau_{p+1}\right); \\
\left[\begin{array}{c}
\sqrt[2q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}\right)+\left(1-\left(1-(\alpha R_{p+1}^{Tl})^{2q}\right)^{\wp_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}\right)\cdot\left(1-\left(1-(\alpha R_{p+1}^{Tl})^{2q}\right)^{\wp_{p+1}}\right),\end{array}} \\
e^{i2\pi\sqrt[2q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}\right)+\left(1-\left(1-\left(\frac{\alpha J_{p+1}^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}\right)\cdot\left(1-\left(1-\left(\frac{\alpha J_{p+1}^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_{p+1}}\right),\end{array}}} \\
\sqrt[2q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}\right)+\left(1-\left(1-(\alpha R_{p+1}^{Tu})^{2q}\right)^{\wp_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}\right)\cdot\left(1-\left(1-(\alpha R_{p+1}^{Tu})^{2q}\right)^{\wp_{p+1}}\right)\end{array}} \\
e^{i2\pi\sqrt[2q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}\right)+\left(1-\left(1-\left(\frac{\alpha J_{p+1}^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}\right)\cdot\left(1-\left(1-\left(\frac{\alpha J_{p+1}^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_{p+1}}\right)\end{array}}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt[q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}\right)+\left(1-\left(1-(\beta R_{p+1}^{Il})^{q}\right)^{\wp_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}\right)\cdot\left(1-\left(1-(\beta R_{p+1}^{Il})^{q}\right)^{\wp_{p+1}}\right),\end{array}} \\
e^{i2\pi\sqrt[q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}\right)+\left(1-\left(1-\left(\frac{\beta J_{p+1}^{Il}}{2\pi}\right)^{q}\right)^{\xi_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}\right)\cdot\left(1-\left(1-\left(\frac{\beta J_{p+1}^{Il}}{2\pi}\right)^{q}\right)^{\xi_{p+1}}\right),\end{array}}} \\
\sqrt[q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}\right)+\left(1-\left(1-(\beta R_{p+1}^{Iu})^{q}\right)^{\wp_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}\right)\cdot\left(1-\left(1-(\beta R_{p+1}^{Iu})^{q}\right)^{\wp_{p+1}}\right)\end{array}} \\
e^{i2\pi\sqrt[q]{\begin{array}{l}\bigoplus_{k=1}^{p}\left(1-\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}\right)+\left(1-\left(1-\left(\frac{\beta J_{p+1}^{Iu}}{2\pi}\right)^{q}\right)^{\xi_{p+1}}\right)\\ -\odot_{k=1}^{p}\left(1-\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}\right)\cdot\left(1-\left(1-\left(\frac{\beta J_{p+1}^{Iu}}{2\pi}\right)^{q}\right)^{\xi_{p+1}}\right)\end{array}}}
\end{array}\right], \\
\left[\begin{array}{c}
\odot_{k=1}^{p}(\gamma R_k^{Fl})^{\wp_k}\cdot(\gamma R_{p+1}^{Fl})^{\wp_{p+1}} e^{i2\pi\odot_{k=1}^{p}\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}\cdot\left(\frac{\gamma J_{p+1}^{Fl}}{2\pi}\right)^{\xi_{p+1}}}, \\
\odot_{k=1}^{p}(\gamma R_k^{Fu})^{\wp_k}\cdot(\gamma R_{p+1}^{Fu})^{\wp_{p+1}} e^{i2\pi\odot_{k=1}^{p}\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}\cdot\left(\frac{\gamma J_{p+1}^{Fu}}{2\pi}\right)^{\xi_{p+1}}}
\end{array}\right]
\end{array}\right]
$$

$$
=\left[\begin{array}{c}
\left(\bigoplus_{k=1}^{p+1}\wp_k\vartheta_k, \bigoplus_{k=1}^{p+1}\wp_k\tau_k\right); \\
\left[\begin{array}{c}
\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}}}, \\
\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}}}, \\
\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\odot_{k=1}^{p+1}(\gamma R_k^{Fl})^{\wp_k} e^{i2\pi\odot_{k=1}^{p+1}\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}}, \odot_{k=1}^{p+1}(\gamma R_k^{Fu})^{\wp_k} e^{i2\pi\odot_{k=1}^{p+1}\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}}\right]
\end{array}\right].
$$

Appendix B. Proof of Theorem 4.2

Given that $(\vartheta_k,\tau_k)=(\vartheta,\tau)$, $[\alpha R_k^{Tl},\alpha R_k^{Tu}]=[\alpha R^{Tl},\alpha R^{Tu}]$, $[\beta R_k^{Il},\beta R_k^{Iu}]=[\beta R^{Il},\beta R^{Iu}]$ and $[\gamma R_k^{Fl},\gamma R_k^{Fu}]=[\gamma R^{Fl},\gamma R^{Fu}]$, $[\alpha J_k^{Tl},\alpha J_k^{Tu}]=[\alpha J^{Tl},\alpha J^{Tu}]$, $[\beta J_k^{Il},\beta J_k^{Iu}]=[\beta J^{Il},\beta J^{Iu}]$ and $[\gamma J_k^{Fl},\gamma J_k^{Fu}]=[\gamma J^{Fl},\gamma J^{Fu}]$, for $k=1,2,\ldots,m$ and $\bigoplus_{k=1}^m \wp_k=1$. Now, q-CDNNIVWA$(\widetilde{\Phi}_1,\widetilde{\Phi}_2,\ldots,\widetilde{\Phi}_m)$

$[\alpha R_k^{Tl},\alpha R_k^{Tu}]=[\alpha R^{Tl},\alpha R^{Tu}]$, $[\beta R_k^{Il},\beta R_k^{Iu}]=[\beta R^{Il},\beta R^{Iu}]$ and $[\gamma R_k^{Fl},\gamma R_k^{Fu}]=[\gamma R^{Fl},\gamma R^{Fu}]$, $[\alpha J_k^{Tl},\alpha J_k^{Tu}]=[\alpha J^{Tl},\alpha J^{Tu}]$, $[\beta J_k^{Il},\beta J_k^{Iu}]=[\beta J^{Il},\beta J^{Iu}]$ and $[\gamma J_k^{Fl},\gamma J_k^{Fu}]=[\gamma J^{Fl},\gamma J^{Fu}]$, for $k=1,2,\ldots,m$ and $\bigoplus_{k=1}^m \wp_k=1$. Now, q-CDNNIVWA$(\widetilde{\Phi}_1,\widetilde{\Phi}_2,\ldots,\widetilde{\Phi}_m)$.

$$
= \begin{bmatrix}
\left(\bigoplus_{k=1}^m \wp_k\vartheta_k, \bigoplus_{k=1}^m \wp_k\tau_k\right); \\
\begin{bmatrix}
\sqrt[2q]{1-\odot_{k=1}^m\left(1-(\alpha R_k^{Tl})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^{2q}\right)^{\xi_k}}}, \\
\sqrt[2q]{1-\odot_{k=1}^m\left(1-(\alpha R_k^{Tu})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^m\left(1-\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^{2q}\right)^{\xi_k}}}
\end{bmatrix}, \\
\begin{bmatrix}
\sqrt[q]{1-\odot_{k=1}^m\left(1-(\beta R_k^{Il})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\frac{\beta J_k^{Il}}{2\pi}\right)^{q}\right)^{\xi_k}}}, \\
\sqrt[q]{1-\odot_{k=1}^m\left(1-(\beta R_k^{Iu})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^m\left(1-\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^{q}\right)^{\xi_k}}}
\end{bmatrix}, \\
\left[\odot_{k=1}^m(\gamma R_k^{Fl})^{\wp_k} e^{i2\pi\odot_{k=1}^m\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{\xi_k}}, \odot_{k=1}^m(\gamma R_k^{Fu})^{\wp_k} e^{i2\pi\odot_{k=1}^m\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{\xi_k}}\right]
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\left(\vartheta\bigoplus_{k=1}^m \wp_k, \tau\bigoplus_{k=1}^m \wp_k\right); \\
\begin{bmatrix}
\sqrt[2q]{1-\left(1-(\alpha R^{Tl})^{2q}\right)^{\bigoplus_{k=1}^m\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\left(1-\left(\frac{\alpha J^{Tl}}{2\pi}\right)^{2q}\right)^{\bigoplus_{k=1}^m\xi_k}}}, \\
\sqrt[2q]{1-\left(1-(\alpha R^{Tu})^{2q}\right)^{\bigoplus_{k=1}^m\wp_k}}\, e^{i2\pi \sqrt[2q]{1\left(1-\left(\frac{\alpha J^{Tu}}{2\pi}\right)^{2q}\right)^{\bigoplus_{k=1}^m\xi_k}}}
\end{bmatrix}, \\
\begin{bmatrix}
\sqrt[q]{1-\left(1-(\beta R^{Il})^{q}\right)^{\bigoplus_{k=1}^m\wp_k}}\, e^{i2\pi \sqrt[q]{1-\left(1-\left(\frac{\beta J^{Il}}{2\pi}\right)^{q}\right)^{\bigoplus_{k=1}^m\xi_k}}}, \\
\sqrt[q]{1-\left(1-(\beta R^{Iu})^{q}\right)^{\bigoplus_{k=1}^m\wp_k}}\, e^{i2\pi \sqrt[q]{1\left(1-\left(\frac{\beta J^{Iu}}{2\pi}\right)^{q}\right)^{\bigoplus_{k=1}^m\xi_k}}}
\end{bmatrix}, \\
\left[(\gamma R^{Fl})^{\bigoplus_{k=1}^m\wp_k} e^{i2\pi\left(\frac{\gamma J^{Fl}}{2\pi}\right)^{\bigoplus_{k=1}^m\xi_k}}, (\gamma R^{Fu})^{\bigoplus_{k=1}^m\wp_k} e^{i2\pi\left(\frac{\gamma J^{Fu}}{2\pi}\right)^{\bigoplus_{k=1}^m\xi_k}}\right]
\end{bmatrix}
$$

$$
= \begin{bmatrix}
(\vartheta,\tau); \\
\begin{bmatrix}
\sqrt[2q]{1-\left(1-(\alpha R^{Tl})^{2q}\right)}\, e^{i2\pi \sqrt[2q]{1-\left(1-\left(\frac{\alpha J^{Tl}}{2\pi}\right)^{2q}\right)}}, \\
\sqrt[2q]{1-\left(1-(\alpha R^{Tu})^{2q}\right)}\, e^{i2\pi \sqrt[2q]{1-\left(1-\left(\frac{\alpha J^{Tu}}{2\pi}\right)^{2q}\right)}}
\end{bmatrix}, \\
\begin{bmatrix}
\sqrt[q]{1-\left(1-(\beta R^{Il})^{q}\right)}\, e^{i2\pi \sqrt[q]{1-\left(1-\left(\frac{\beta J^{Il}}{2\pi}\right)^{q}\right)}}, \\
\sqrt[q]{1-\left(1-(\beta R^{Iu})^{q}\right)}\, e^{i2\pi \sqrt[q]{1-\left(1-\left(\frac{\beta J^{Iu}}{2\pi}\right)^{q}\right)}}
\end{bmatrix}, \\
\left[(\gamma R^{Fl}) e^{i2\pi\left(\frac{\gamma J^{Fl}}{2\pi}\right)}, (\gamma R^{Fu}) e^{i2\pi\left(\frac{\gamma J^{Fu}}{2\pi}\right)}\right]
\end{bmatrix}
$$

$= \widetilde{\Phi}.$

Appendix C. Proof of Theorem 4.3

Since, $\underline{\alpha R^{Tl}} = \min \alpha R^{T-}_{kj}$, $\overline{\alpha R^{Tl}} = \max \alpha R^{T-}_{kj}$ $\underline{\alpha R^{Tu}} = \min \alpha R^{T+}_{kj}$, $\overline{\alpha R^{Tu}} = \max \alpha R^{T+}_{kj}$ and $\underline{\alpha R^{Tl}} \leq \alpha R^{T-}_{kj} \leq \overline{\alpha R^{Tl}}$ and $\underline{\alpha R^{Tu}} \leq \alpha R^{T+}_{kj} \leq \overline{\alpha R^{Tu}}$. We have,

$$\begin{aligned}
\underline{\alpha R^{Tl}} + \underline{\alpha R^{Tu}} &= \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\underline{\alpha R^{Tl}})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\alpha J^{Tl}}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \\
&+ \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\underline{\alpha R^{Tu}})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\alpha J^{Tu}}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \\
&\leq \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R^{T-}_{kj})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J^{T-}_{kj}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \\
&+ \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R^{T+}_{kj})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J^{T+}_{kj}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \\
&\leq \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\overline{\alpha R^{Tl}})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\alpha J^{Tl}}}{2\pi}\right)^{2q}\right)^{\xi_k}}} \\
&+ \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\overline{\alpha R^{Tu}})^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\alpha J^{Tu}}}{2\pi}\right)^{2q}\right)^{\xi_i}}} \\
&= \overline{\alpha R^{Tl}} + \overline{\alpha R^{Tu}}.
\end{aligned}$$

Since, $\underline{\beta R^{Il}} = \min \beta R^{I-}_{kj}$, $\overline{\beta R^{Il}} = \max \beta R^{I-}_{kj}$ $\underline{\beta R^{Iu}} = \min \beta R^{I+}_{kj}$, $\overline{\beta R^{Iu}} = \max \beta R^{I+}_{kj}$ and $\underline{\beta R^{Il}} \leq \beta R^{I-}_{kj} \leq \overline{\beta R^{Il}}$ and $\underline{\beta R^{Iu}} \leq \beta R^{I+}_{kj} \leq \overline{\beta R^{Iu}}$. We have,

$$\begin{aligned}
\underline{\beta R^{Il}} + \underline{\beta R^{Iu}} &= \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\underline{\beta R^{Il}})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\beta J^{Il}}}{2\pi}\right)^{q}\right)^{\xi_k}}} \\
&+ \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\underline{\beta R^{Iu}})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\beta J^{Iu}}}{2\pi}\right)^{q}\right)^{\xi_k}}} \\
&\leq \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R^{I-}_{kj})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J^{I-}_{kj}}{2\pi}\right)^{q}\right)^{\xi_k}}} \\
&+ \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R^{I+}_{kj})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J^{I+}_{kj}}{2\pi}\right)^{q}\right)^{\xi_k}}} \\
&\leq \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\overline{\beta R^{Il}})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\beta J^{Il}}}{2\pi}\right)^{q}\right)^{\xi_k}}} \\
&+ \sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\overline{\beta R^{Iu}})^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\beta J^{Iu}}}{2\pi}\right)^{q}\right)^{\xi_i}}} \\
&= \overline{\beta R^{Il}} + \overline{\beta R^{Iu}}.
\end{aligned}$$

Since, $\underline{\gamma R^{Fl}} = \min \gamma R^{F-}_{kj}$, $\overline{\gamma R^{Fl}} = \max \gamma R^{F-}_{kj}$ $\underline{\gamma R^{Fu}} = \min \gamma R^{F+}_{kj}$, $\overline{\gamma R^{Fu}} = \max \gamma R^{F+}_{kj}$ and $\underline{\gamma R^{Fl}} \leq \gamma R^{F-}_{kj} \leq \overline{\gamma R^{Fl}}$ and $\underline{\gamma R^{Fu}} \leq R^{F+}_{kj} \leq \overline{\gamma R^{Fu}}$. We have,

$$\begin{aligned}
\underline{\gamma R^{Fl}} + \underline{\gamma R^{Fu}} &= \odot_{k=1}^{m}(\underline{\gamma R^{Fl}})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\underline{\gamma J^{Fl}}}{2\pi}\right)^{\xi_k}} + \odot_{k=1}^{m}(\underline{\gamma R^{Fu}})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\underline{\gamma J^{Fu}}}{2\pi}\right)^{\xi_k}} \\
&\leq \odot_{k=1}^{m}(\gamma R^{F-}_{kj})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\gamma J^{F-}_{kj}}{2\pi}\right)^{\xi_k}} + \odot_{k=1}^{m}(\gamma R^{F+}_{kj})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\gamma J^{F+}_{kj}}{2\pi}\right)^{\xi_k}} \\
&\leq \odot_{k=1}^{m}(\overline{\gamma R^{Fl}})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\overline{\gamma J^{Fl}}}{2\pi}\right)^{\xi_k}} + \odot_{k=1}^{m}(\overline{\gamma R^{Fu}})^{\wp_k} e^{i2\pi\odot_{k=1}^{m}\left(\frac{\overline{\gamma J^{Fu}}}{2\pi}\right)^{\xi_k}} \\
&= \overline{\gamma R^{Fl}} + \overline{\gamma R^{Fu}}.
\end{aligned}$$

Since, $\underline{\vartheta}=\min\vartheta_{kj}$, $\overline{\vartheta}=\max\vartheta_{kj}$, $\underline{\tau}=\max\tau_{kj}$, $\overline{\tau}=\min\tau_{kj}$ and $\underline{\vartheta}\le\vartheta_{kj}\le\overline{\vartheta}$ and $\overline{\tau}\le\tau_{kj}\le\underline{\tau}$.
Hence, $\bigoplus_{k=1}^{m}\wp_k\underline{\vartheta}\le\bigoplus_{k=1}^{m}\wp_k\vartheta_{kj}\le\bigoplus_{k=1}^{m}\wp_k\overline{\vartheta}$ and $\bigoplus_{k=1}^{m}\wp_k\overline{\tau}\le\bigoplus_{k=1}^{m}\wp_k\tau_{kj}\le\bigoplus_{k=1}^{m}\wp_k\underline{\tau}$.
Therefore,

$$\frac{\bigoplus_{k=1}^{m}\wp_k\underline{\vartheta}}{2}\times\left[\left[\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\underline{\alpha R^{Tl}})^{2q}\right)^{\wp_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\underline{\alpha R^{Tu}})^{2q}\right)^{\wp_k}}\right)^2}{2}-\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\underline{\beta R^{Il}})^{q}\right)^{\wp_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\underline{\beta R^{Iu}})^{q}\right)^{\wp_k}}\right)^2}{2}+2-\frac{\left(\odot_{k=1}^{m}(\overline{\gamma R^{Fl}})^{\wp_k}\right)^2+\left(\odot_{k=1}^{m}(\overline{\gamma R^{Fu}})^{\wp_k}\right)^2}{2}\right]+\left[e^{i2\pi\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\alpha J^{Tl}}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\alpha J^{Tu}}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2}{2}}-e^{i2\pi\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\beta J^{Il}}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\underline{\beta J^{Iu}}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2}{2}}-\left(e^{i2\pi\frac{\left(\odot_{k=1}^{m}\left(\frac{\overline{\gamma J^{Fl}}}{2\pi}\right)^{\xi_k}\right)^2+\left(\odot_{k=1}^{m}\left(\frac{\overline{\gamma J^{Fu}}}{2\pi}\right)^{\xi_k}\right)^2}{2}}\right)\right]\right]$$

$$\le\frac{\bigoplus_{k=1}^{m}\wp_k\vartheta_{kj}}{2}\times\left[\left[\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{kj}^{T^-})^{2q}\right)^{\wp_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{kj}^{T^+})^{2q}\right)^{\wp_k}}\right)^2}{2}-\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{kj}^{I^-})^{q}\right)^{\wp_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{kj}^{I^+})^{q}\right)^{\wp_k}}\right)^2}{2}+2-\frac{\left(\odot_{k=1}^{m}(\gamma R_{kj}^{F^-})^{\wp_k}\right)^2+\left(\odot_{k=1}^{m}(\gamma R_{kj}^{F^+})^{\wp_k}\right)^2}{2}\right]+\left[e^{i2\pi\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{kj}^{T^-}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{kj}^{T^+}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2}{2}}-e^{i2\pi\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{kj}^{I^-}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{kj}^{I^+}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2}{2}}-\left(e^{i2\pi\frac{\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{kj}^{F^-}}{2\pi}\right)^{\xi_k}\right)^2+\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{kj}^{F^+}}{2\pi}\right)^{\xi_k}\right)^2}{2}}\right)\right]\right]$$

$$\le\frac{\bigoplus_{k=1}^{m}\wp_k\overline{\vartheta}}{2}\times\left[\left[\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\overline{\alpha R^{Tl}})^{2q}\right)^{\wp_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\overline{\alpha R^{Tu}})^{2q}\right)^{\wp_k}}\right)^2}{2}-\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\overline{\beta R^{Il}})^{q}\right)^{\wp_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\overline{\beta R^{Iu}})^{q}\right)^{\wp_k}}\right)^2}{2}+2-\frac{\left(\odot_{k=1}^{m}(\underline{\gamma R^{Fl}})^{\wp_k}\right)^2+\left(\odot_{k=1}^{m}(\underline{\gamma R^{Fu}})^{\wp_k}\right)^2}{2}\right]+\left[e^{i2\pi\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\alpha J^{Tl}}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\alpha J^{Tu}}}{2\pi}\right)^{2q}\right)^{\xi_k}}\right)^2}{2}}-e^{i2\pi\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\beta J^{Il}}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\overline{\beta J^{Iu}}}{2\pi}\right)^{q}\right)^{\xi_k}}\right)^2}{2}}-\left(e^{i2\pi\frac{\left(\odot_{k=1}^{m}\left(\frac{\underline{\gamma J^{Fl}}}{2\pi}\right)^{\xi_k}\right)^2+\left(\odot_{k=1}^{m}\left(\frac{\underline{\gamma J^{Fu}}}{2\pi}\right)^{\xi_k}\right)^2}{2}}\right)\right]\right].$$

Hence,

$$\begin{aligned}
&\Big\langle (\underline{\vartheta}, \underline{\tau}); [\underline{\alpha R^{Tl} e^{i2\pi(\alpha J^{Tl})}}, \underline{\alpha R^{Tu} e^{i2\pi(\alpha J^{Tu})}}], [\underline{\beta R^{Il} e^{i2\pi(\beta J^{Il})}}, \underline{\beta R^{Iu} e^{i2\pi(\beta J^{Iu})}}], [\overline{\gamma R^{Fl} e^{i2\pi(\gamma J^{Fl})}}, \overline{\gamma R^{Fu} e^{i2\pi(\gamma J^{Fu})}}] \Big\rangle \\
&\leq q-CDNNIVWA(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m) \\
&\leq \Big\langle (\overline{\vartheta}, \overline{\tau}); [\overline{\alpha R^{Tl} e^{i2\pi(\alpha J^{Tl})}}, \overline{\alpha R^{Tu} e^{i2\pi(\alpha J^{Tu})}}], [\overline{\beta R^{Il} e^{i2\pi(\beta J^{Il})}}, \overline{\beta R^{Iu} e^{i2\pi(\beta J^{Iu})}}], [\underline{\gamma R^{Fl} e^{i2\pi(\gamma J^{Fl})}}, \underline{\gamma R^{Fu} e^{i2\pi(\gamma J^{Fu})}}] \Big\rangle.
\end{aligned}$$

Appendix D. Proof of Theorem 4.4

For any i, $\vartheta_{t_{kj}} \leq \tau_{h_{kj}}$. Therefore, $\bigoplus_{k=1}^{m} \vartheta_{t_{kj}} \leq \bigoplus_{k=1}^{m} \tau_{h_{kj}}$.

For any i, $\left(\alpha R_{t_{kj}}^{Tl}\right)^2 + \left(\alpha R_{t_{kj}}^{Tu}\right)^2 \leq \left(\alpha R_{h_{kj}}^{Tl}\right)^2 + \left(\alpha R_{h_{kj}}^{Tu}\right)^2$.

Therefore, $1-\left(\alpha R_{t_k}^{Tl}\right)^2 + 1-\left(\alpha R_{t_k}^{Tu}\right)^2 \geq 1-\left(\alpha R_{h_k}^{Tl}\right)^2 + 1-\left(\alpha R_{h_k}^{Tu}\right)^2$.

For any i, $\left(\alpha J_{t_{kj}}^{Tl}\right)^2 + \left(\alpha J_{t_{kj}}^{Tu}\right)^2 \leq \left(\alpha J_{h_{kj}}^{Tl}\right)^2 + \left(\alpha J_{h_{kj}}^{Tu}\right)^2$.

Therefore, $1-\left(\alpha J_{t_k}^{Tl}\right)^2 + 1-\left(\alpha J_{t_k}^{Tu}\right)^2 \geq 1-\left(\alpha J_{h_k}^{Tl}\right)^2 + 1-\left(\alpha J_{h_k}^{Tu}\right)^2$.

Hence, $\odot_{k=1}^{m}\left(1-\left(\alpha R_{t_k}^{Tl}\right)^2\right)^{\wp_k} + \odot_{k=1}^{m}\left(1-\left(\alpha R_{t_k}^{Tu}\right)^2\right)^{\wp_k} \geq \odot_{k=1}^{m}\left(1-\left(\alpha R_{h_k}^{Tl}\right)^2\right)^{\wp_k} + \odot_{k=1}^{m}\left(1-\left(\alpha R_{h_k}^{Tu}\right)^2\right)^{\wp_k}$

and $\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\alpha R_{t_k}^{Tl}\right)^{2q}\right)^{\wp_k}} + \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\alpha R_{t_k}^{Tu}\right)^{2q}\right)^{\wp_k}} \leq \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\alpha R_{h_k}^{Tl}\right)^{2q}\right)^{\wp_k}} + \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\alpha R_{h_k}^{Tu}\right)^{2q}\right)^{\wp_k}}$.

and $e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{t_k}^{Tl}}{2\pi}\right)^{2q}\right)^{\wp_k}} + \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{t_k}^{Tu}}{2\pi}\right)^{2q}\right)^{\wp_k}}} \leq e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{h_k}^{Tl}}{2\pi}\right)^{2q}\right)^{\wp_k}} + \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{h_k}^{Tu}}{2\pi}\right)^{2q}\right)^{\wp_k}}}$.

For any i, $\left(\beta R_{t_{kj}}^{Il}\right)^2 + \left(\beta R_{t_{kj}}^{Iu}\right)^2 \leq \left(\beta R_{h_{kj}}^{Il}\right)^2 + \left(\beta R_{h_{kj}}^{Iu}\right)^2$.

Therefore, $1-\left(\beta R_{t_k}^{Il}\right)^2 + 1-\left(\beta R_{t_k}^{Iu}\right)^2 \geq 1-\left(\beta R_{h_k}^{Il}\right)^2 + 1-\left(\beta R_{h_k}^{Iu}\right)^2$.

For any i, $\left(\beta J_{t_{kj}}^{Il}\right)^2 + \left(\beta J_{t_{kj}}^{Iu}\right)^2 \leq \left(\beta J_{h_{kj}}^{Il}\right)^2 + \left(\beta J_{h_{kj}}^{Iu}\right)^2$.

Therefore, $1-\left(\beta J_{t_k}^{Il}\right)^2 + 1-\left(\beta J_{t_k}^{Iu}\right)^2 \geq 1-\left(\beta J_{h_k}^{Il}\right)^2 + 1-\left(\beta J_{h_k}^{Iu}\right)^2$.

Hence, $\odot_{k=1}^{m}\left(1-\left(\beta R_{t_k}^{Il}\right)^2\right)^{\wp_k} + \odot_{k=1}^{m}\left(1-\left(\beta R_{t_k}^{Iu}\right)^2\right)^{\wp_k} \geq \odot_{k=1}^{m}\left(1-\left(\beta R_{h_k}^{Il}\right)^2\right)^{\wp_k} + \odot_{k=1}^{m}\left(1-\left(\beta R_{h_k}^{Iu}\right)^2\right)^{\wp_k}$

and $\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\beta R_{t_k}^{Il}\right)^{q}\right)^{\wp_k}} + \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\beta R_{t_k}^{Iu}\right)^{q}\right)^{\wp_k}} \leq \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\beta R_{h_k}^{Il}\right)^{q}\right)^{\wp_k}} + \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\beta R_{h_k}^{Iu}\right)^{q}\right)^{\wp_k}}$.

and $e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{t_k}^{Il}}{2\pi}\right)^{q}\right)^{\wp_k}} + \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{t_k}^{Iu}}{2\pi}\right)^{q}\right)^{\wp_k}}} \leq e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{h_k}^{Il}}{2\pi}\right)^{q}\right)^{\wp_k}} + \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{h_k}^{Iu}}{2\pi}\right)^{q}\right)^{\wp_k}}}$.

For any i, $\left(\gamma R_{t_{kj}}^{Fl}\right)^2 + \left(\gamma R_{t_{kj}}^{Fu}\right)^2 \geq \left(\gamma R_{h_{kj}}^{Fl}\right)^2 + \left(\gamma R_{h_{kj}}^{Fu}\right)^2$.

Therefore, $2-\frac{\left(\odot_{k=1}^{m} R_{t_{kj}}^{Fl}\right)^2 + \left(\odot_{k=1}^{m} R_{t_{kj}}^{Fu}\right)^2}{2} \leq 2-\frac{\left(\odot_{k=1}^{m} R_{h_{kj}}^{Fl}\right)^2 + \left(\odot_{k=1}^{m} R_{h_{kj}}^{Fu}\right)^2}{2}$.

and $-e^{i2\pi \frac{\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{t_{kj}}^{Fl}}{2\pi}\right)\right)^2 + \left(\odot_{k=1}^{m}\left(\frac{\gamma J_{t_{kj}}^{Fu}}{2\pi}\right)\right)^2}{2}} \leq -e^{i2\pi \frac{\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{h_{kj}}^{Fl}}{2\pi}\right)\right)^2 + \left(\odot_{k=1}^{m}\left(\frac{\gamma J_{h_{kj}}^{Fu}}{2\pi}\right)\right)^2}{2}}$.

Hence

$$\frac{\bigoplus_{k=1}^{m} \vartheta_{tkj}}{2} \times \left[\begin{array}{l}
\left[\begin{array}{c}
\frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{tk}^{T^-})^{2q}\right)^{\wp_k}}\right)^2 + \left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{tk}^{T^+})^{2q}\right)^{\wp_k}}\right)^2}{2} \\
-\frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{tk}^{I^-})^{q}\right)^{\wp_k}}\right)^2 + \left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{tk}^{I^+})^{q}\right)^{\wp_k}}\right)^2}{2} \\
+2-\frac{\left(\odot_{k=1}^{m}(\gamma R_{tkj}^{F^-})\right)^2 + \left(\odot_{k=1}^{m}(\gamma R_{tkj}^{F^+})\right)^2}{2}
\end{array}\right] \\
+\left[\begin{array}{c}
e^{i2\pi \frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{t_k}^{Tl}}{2\pi}\right)^{2q}\right)^{\wp_k}}\right)^2 + \left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{t_k}^{Tu}}{2\pi}\right)^{2q}\right)^{\wp_k}}\right)^2}{2}} \\
-e^{i2\pi \frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{t_k}^{Il}}{2\pi}\right)^{q}\right)^{\wp_k}}\right)^2 + \left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{t_k}^{Iu}}{2\pi}\right)^{q}\right)^{\wp_k}}\right)^2}{2}} \\
-e^{i2\pi \frac{\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{t_{kj}}^{Fl}}{2\pi}\right)\right)^2 + \left(\odot_{k=1}^{m}\left(\frac{\gamma J_{t_{kj}}^{Fu}}{2\pi}\right)\right)^2}{2}}
\end{array}\right]
\end{array}\right]$$

$$
\leq \frac{\bigoplus_{k=1}^{m} \vartheta_{hkj}}{2} \times \left[\begin{array}{l}
\left[\begin{array}{c}
\dfrac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{hk}^{T^-})^{2q}\right)^{\wp_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-(\alpha R_{hk}^{T^+})^{2q}\right)^{\wp_k}}\right)^2}{2} \\
\dfrac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{hk}^{I^-})^{q}\right)^{\wp_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-(\beta R_{hk}^{I^+})^{q}\right)^{\wp_k}}\right)^2}{2} \\
+2-\dfrac{\left(\odot_{k=1}^{m}(\gamma R_{hkj}^{F^-})\right)^2+\left(\odot_{k=1}^{m}(\gamma R_{hkj}^{F^+})\right)^2}{2}
\end{array}\right] \\
+\left[\begin{array}{l}
e^{i2\pi \frac{\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{h_k}^{Tl}}{2\pi}\right)^{2q}\right)^{\wp_k}}\right)^2+\left(\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\alpha J_{h_k}^{Tu}}{2\pi}\right)^{2q}\right)^{\wp_k}}\right)^2}{2}} \\
-e^{i2\pi \frac{\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{h_k}^{Il}}{2\pi}\right)^{q}\right)^{\wp_k}}\right)^2+\left(\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\frac{\beta J_{h_k}^{Iu}}{2\pi}\right)^{q}\right)^{\wp_k}}\right)^2}{2}} \\
-e^{i2\pi \frac{\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{h_{kj}}^{Fl}}{2\pi}\right)\right)^2+\left(\odot_{k=1}^{m}\left(\frac{\gamma J_{h_{kj}}^{Fu}}{2\pi}\right)\right)^2}{2}}
\end{array}\right]
\end{array}\right].
$$

Hence, q-CDNNIVWA $\left(\widetilde{\Phi}_1, \widetilde{\Phi}_2, \ldots, \widetilde{\Phi}_m\right) \leq q$-CDNNIVWA $\left(\widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_m\right)$.

Appendix E. Proof of Theorem 4.7

To prove that, $\bigoplus_{k=1}^{m} \wp_k \widetilde{\Phi}_k^q =$

$$
\left[\begin{array}{c}
\left(\left(\bigoplus_{k=1}^{m} \wp_k \vartheta_k^q\right), \left(\bigoplus_{k=1}^{m} \wp_k \tau_k^q\right)\right); \\
\left[\begin{array}{l}
\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left((\alpha R_k^{Tl})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}, \\
\sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left((\alpha R_k^{Tu})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[2q]{1-\odot_{k=1}^{m}\left(1-\left(\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{l}
\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left((\beta R_k^{Il})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}, \\
\sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left((\beta R_k^{Iu})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi \sqrt[q]{1-\odot_{k=1}^{m}\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{l}
\odot_{k=1}^{m}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi \odot_{k=1}^{m}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}, \\
\odot_{k=1}^{m}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fu})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi \odot_{k=1}^{m}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}
\end{array}\right]
\end{array}\right].
$$

Put $m = 2$, $\wp_1\Phi_A \bigoplus \wp_2\Phi_B$

$$
=\left[\begin{array}{c}
\left(\wp_1\vartheta_1^q+\wp_2\vartheta_2^q,\wp_1\tau_1^q+\wp_2\tau_2^q\right); \\
\left[\begin{array}{l}
\sqrt[2q]{\begin{array}{l}\left(\sqrt[2q]{1-\left(1-\left((\alpha R_1^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}+\left(\sqrt[2q]{1-\left(1-\left((\alpha R_2^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}, \\ -\left(\sqrt[2q]{1-\left(1-\left((\alpha R_1^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left((\alpha R_2^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\end{array}} \\
e^{i2\pi\sqrt[2q]{\begin{array}{l}\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}+\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_2^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}, \\ -\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_2^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\end{array}}}, \\
\sqrt[2q]{\begin{array}{l}\left(\sqrt[2q]{1-\left(1-\left((\alpha R_1^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}+\left(\sqrt[2q]{1-\left(1-\left((\alpha R_2^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q} \\ -\left(\sqrt[2q]{1-\left(1-\left((\alpha R_1^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left((\alpha R_2^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\end{array}} \\
e^{i2\pi\sqrt[2q]{\begin{array}{l}\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}+\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_2^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q} \\ -\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_2^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\end{array}}}
\end{array}\right], \\
\left[\begin{array}{l}
\sqrt[q]{\begin{array}{l}\left(\sqrt[q]{1-\left(1-\left((\beta R_1^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}+\left(\sqrt[q]{1-\left(1-\left((\beta R_2^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}, \\ -\left(\sqrt[q]{1-\left(1-\left((\beta R_1^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left((\beta R_2^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\end{array}} \\
e^{i2\pi\sqrt[q]{\begin{array}{l}\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_1^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}+\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_2^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}, \\ -\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_1^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_2^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\end{array}}}, \\
\sqrt[q]{\begin{array}{l}\left(\sqrt[q]{1-\left(1-\left((\beta R_1^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q}+\left(\sqrt[q]{1-\left(1-\left((\beta R_2^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q} \\ -\left(\sqrt[q]{1-\left(1-\left((\beta R_1^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left((\beta R_2^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\end{array}} \\
e^{i2\pi\sqrt[q]{\begin{array}{l}\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}+\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_2^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q} \\ -\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_2^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\end{array}}}
\end{array}\right], \\
\left[\begin{array}{l}
\left(\sqrt[2q]{1-\left(1-(\gamma R_1^{Fl})^{2q}\right)^q}\right)^{\wp_1}\cdot e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_1^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}\cdot \\
\left(\sqrt[2q]{1-\left(1-(\gamma R_2^{Fl})^{2q}\right)^q}\right)^{\wp_1} e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_2^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}, \\
\left(\sqrt[2q]{1-\left(1-(\gamma R_1^{Fu})^{2q}\right)^q}\right)^{\wp_1}\cdot e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_1^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}\cdot \\
\left(\sqrt[2q]{1-\left(1-(\gamma R_2^{Fu})^{2q}\right)^q}\right)^{\wp_1} e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_2^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}
\end{array}\right]
\end{array}\right]
$$

$$
= \left[\begin{array}{c}
\left(\bigoplus_{k=1}^{2}\wp_k\vartheta_k^q, \bigoplus_{k=1}^{2}\wp_k\tau_k^q\right); \\
\left[\begin{array}{c}
\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left((\alpha R_1^{Tl})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left(\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}} \\
\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left((\alpha R_1^{Tu})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{2}\left(1-\left(\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left((\beta R_1^{Il})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left(\left(\frac{\beta J_1^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}} \\
\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left((\beta R_1^{Iu})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{2}\left(1-\left(\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{c}
\odot_{k=1}^{2}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{2}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}} \\
\odot_{k=1}^{2}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fu})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{2}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}
\end{array}\right]
\end{array}\right].
$$

In general,

$$
\bigoplus_{k=1}^{p}\wp_k\widetilde{\boldsymbol{\Phi}}_k^q = \left[\begin{array}{c}
\left(\bigoplus_{k=1}^{p}\wp_k\vartheta_k^q, \bigoplus_{k=1}^{p}\wp_k\tau_k^q\right); \\
\left[\begin{array}{c}
\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_1^{Tl})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}, \\
\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_1^{Tu})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_1^{Il})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_1^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}, \\
\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_1^{Iu})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}
\end{array}\right], \\
\left[\begin{array}{c}
\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}, \\
\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}
\end{array}\right]
\end{array}\right].
$$

If $m = p+1$, then $\bigoplus_{k=1}^{p}\wp_k\widetilde{\boldsymbol{\Phi}}_k^q + \wp_{p+1}\widetilde{\boldsymbol{\Phi}}_{p+1}^q = \bigoplus_{k=1}^{p+1}\wp_k\widetilde{\boldsymbol{\Phi}}_k^q$.

Now, $\bigoplus_{k=1}^{p} \wp_k \widetilde{\boldsymbol{\Phi}}_k^q + \wp_{p+1}\widetilde{\boldsymbol{\Phi}}_{p+1}^q = \wp_1\widetilde{\boldsymbol{\Phi}}_1^q \bigoplus \wp_2 \widetilde{\boldsymbol{\Phi}}_2^q \bigoplus \dots \bigoplus \wp_p\widetilde{\boldsymbol{\Phi}}_p^q \bigoplus \wp_{p+1}\widetilde{\boldsymbol{\Phi}}_{p+1}^q$

$$
= \left[\begin{array}{c}
\left(\bigoplus_{k=1}^{p} \wp_k \vartheta_k^q + \wp_{p+1}\vartheta_{p+1}^q, \bigoplus_{k=1}^{p} \wp_k \tau_k^q + \wp_{p+1}\tau_{p+1}^q\right); \\
\left[\begin{array}{c}
\sqrt[2q]{\begin{array}{c}\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_k^{Tl})^q\right)^{2q}\right)^{\wp_k}}\right)^{2q} + \left(\sqrt[2q]{1-\left(1-\left((\alpha R_{p+1}^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}, \\ -\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_k^{Tl})^q\right)^{2q}\right)^{\wp_k}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left((\alpha R_{p+1}^{Tl})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\end{array}} \\
e^{i2\pi \sqrt[2q]{\begin{array}{c}\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{2q} + \left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_{p+1}^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}, \\ -\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_{p+1}^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\end{array}}} \\
\sqrt[2q]{\begin{array}{c}\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_k^{Tu})^q\right)^{2q}\right)^{\wp_k}}\right)^{2q} + \left(\sqrt[2q]{1-\left(1-\left((\alpha R_{p+1}^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q} \\ -\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left((\alpha R_k^{Tu})^q\right)^{2q}\right)^{\wp_k}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left((\alpha R_{p+1}^{Tu})^q\right)^{2q}\right)^{\wp_1}}\right)^{2q}\end{array}} \\
e^{i2\pi \sqrt[2q]{\begin{array}{c}\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{2q} + \left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_{p+1}^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q} \\ -\left(\sqrt[2q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{2q}\cdot\left(\sqrt[2q]{1-\left(1-\left(\left(\frac{\alpha J_{p+1}^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_1}}\right)^{2q}\end{array}}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt[q]{\begin{array}{c}\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_k^{Il})^q\right)^{q}\right)^{\wp_k}}\right)^{q} + \left(\sqrt[q]{1-\left(1-\left((\beta R_{p+1}^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}, \\ -\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_k^{Il})^q\right)^{q}\right)^{\wp_k}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left((\beta R_{p+1}^{Il})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\end{array}} \\
e^{i2\pi \sqrt[q]{\begin{array}{c}\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{q} + \left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_{p+1}^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}, \\ -\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_{p+1}^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\end{array}}} \\
\sqrt[q]{\begin{array}{c}\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_k^{Iu})^q\right)^{q}\right)^{\wp_k}}\right)^{q} + \left(\sqrt[q]{1-\left(1-\left((\beta R_{p+1}^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q} \\ -\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left((\beta R_k^{Iu})^q\right)^{q}\right)^{\wp_k}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left((\beta R_{p+1}^{Iu})^q\right)^{q}\right)^{\wp_1}}\right)^{q}\end{array}} \\
e^{i2\pi \sqrt[q]{\begin{array}{c}\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{q} + \left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_{p+1}^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q} \\ -\left(\sqrt[q]{1-\odot_{k=1}^{p}\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{q}\cdot\left(\sqrt[q]{1-\left(1-\left(\left(\frac{\beta J_{p+1}^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_1}}\right)^{q}\end{array}}}
\end{array}\right], \\
\left[\begin{array}{c}
\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k}\cdot e^{i2\pi\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}\cdot \\
\left(\sqrt[2q]{1-\left(1-(\gamma R_{p+1}^{Fl})^{2q}\right)^q}\right)^{\wp_1} e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_{p+1}^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}, \\
\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fu})^{2q}\right)^q}\right)^{\wp_k}\cdot e^{i2\pi\odot_{k=1}^{p}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}\cdot \\
\left(\sqrt[2q]{1-\left(1-(\gamma R_{p+1}^{Fu})^{2q}\right)^q}\right)^{\wp_1} e^{i2\pi\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_{p+1}^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_1}}
\end{array}\right]
\end{array}\right]
$$

$$\bigoplus_{k=1}^{p+1}\wp_k\widetilde{\Phi}_k^q = \left[\begin{array}{c} \left(\bigoplus_{k=1}^{p+1}\wp_k\vartheta_k^q, \bigoplus_{k=1}^{p+1}\wp_k\tau_k^q\right); \\ \left[\begin{array}{c} \sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left((\alpha R_1^{Tl})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\alpha J_1^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}}, \\ \sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left((\alpha R_1^{Tu})^q\right)^{2q}\right)^{\wp_k}}\, e^{i2\pi 1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\alpha J_1^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}} \end{array}\right], \\ \left[\begin{array}{c} \sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left((\beta R_1^{Il})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\beta J_1^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}}, \\ \sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left((\beta R_1^{Iu})^q\right)^{q}\right)^{\wp_k}}\, e^{i2\pi 1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\beta J_1^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}} \end{array}\right], \\ \left[\begin{array}{c} \odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}}, \\ \odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k} e^{i2\pi\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}} \end{array}\right] \end{array}\right].$$

$\left(\bigoplus_{k=1}^{p+1}\wp_k\widetilde{\Phi}_k^q\right)^{1/q}$ can be written as follows:

$$\left[\begin{array}{c} \left(\left(\bigoplus_{k=1}^{p+1}\wp_k\vartheta_k^q\right)^{1/q}, \left(\bigoplus_{k=1}^{p+1}\wp_k\tau_k^q\right)^{1/q}\right); \\ \left[\begin{array}{c} \left(\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left((\alpha R_k^{Tl})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\alpha J_k^{Tl}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}}, \\ \left(\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left((\alpha R_k^{Tu})^q\right)^{2q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[2q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\alpha J_k^{Tu}}{2\pi}\right)^q\right)^{2q}\right)^{\xi_k}}\right)^{1/q}} \end{array}\right], \\ \left[\begin{array}{c} \left(\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left((\beta R_k^{Il})^q\right)^{q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\beta J_k^{Il}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{1/q}}, \\ \left(\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left((\beta R_k^{Iu})^q\right)^{q}\right)^{\wp_k}}\right)^{1/q} e^{i2\pi\left(\sqrt[q]{1-\odot_{k=1}^{p+1}\left(1-\left(\left(\frac{\beta J_k^{Iu}}{2\pi}\right)^q\right)^{q}\right)^{\xi_k}}\right)^{1/q}} \end{array}\right], \\ \left[\begin{array}{c} \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fl})^{2q}\right)^q}\right)^{\wp_k}\right)^2\right)^{1/q}} \\ e^{i2\pi\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fl}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^2\right)^{1/q}}}, \\ \sqrt[2q]{1-\left(1-\left(\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-(\gamma R_k^{Fu})^{2q}\right)^q}\right)^{\wp_k}\right)^2\right)^{1/q}} \\ e^{i2\pi\sqrt[2q]{1-\left(1-\left(\odot_{k=1}^{p+1}\left(\sqrt[2q]{1-\left(1-\left(\frac{\gamma J_k^{Fu}}{2\pi}\right)^{2q}\right)^q}\right)^{\xi_k}\right)^2\right)^{1/q}}}, \end{array}\right] \end{array}\right].$$

Data availability

No data was used for the research described in the article.

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