mathematics

Article

Generalized Open Sets and Closure Operators via Point-to-Neighborhood Assignments

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Abstract

We equip a topological space (X, τ) with a function $\mathfrak{a} : X \to \tau$ satisfying the single axiom $x \in \mathfrak{a}(x)$. The resulting triple $(X, \tau, \mathfrak{a})$, which we call an *aura topological space*, provides a point-to-open-set assignment that differs from all existing auxiliary structures in topology—ideals, filters, grills, primals, and the various non-classical frameworks based on fuzzy, soft, or neutrosophic sets. The aura-closure operator $\mathrm{cl}_{\mathfrak{a}}(A) = \{x \in X : \mathfrak{a}(x) \cap A \neq \varnothing\}$ is shown to be an additive Čech closure operator; it satisfies extensivity, monotonicity, and finite additivity, but idempotency fails in general. Iterating $\mathrm{cl}_{\mathfrak{a}}$ transfinitely yields a Kuratowski closure whose topology $\tau_{\mathfrak{a}}^{\infty}$ satisfies $\tau_{\mathfrak{a}}^{\infty} = \tau_{\mathfrak{a}} \subseteq \tau$, where $\tau_{\mathfrak{a}}$ is the collection of all $\mathfrak{a}$-open sets. We introduce $\mathfrak{a}$-semi-open, $\mathfrak{a}$-pre-open, $\mathfrak{a}$-α-open, and $\mathfrak{a}$-β-open sets, determine the complete hierarchy among these classes and their classical counterparts, and separate all non-coinciding classes by counterexamples on finite spaces as well as on the real line. The notions of $\mathfrak{a}$-convergence of sequences and the corresponding continuity notions and their decompositions are studied. Separation axioms $\mathfrak{a}$-T_i $(i = 0, 1, 2)$ are introduced, and it is proved that $\mathfrak{a}$-T_1 and $\mathfrak{a}$-T_2 are equivalent. A detailed comparison with ideals, filters, grills, and primals highlights the distinctive features of the aura framework.

Keywords: aura topological space; scope function; point-to-neighborhood assignment; aura-closure operator; generalized open sets; Čech closure; Kuratowski closure; separation axioms; $\mathfrak{a}$-convergence

MSC: 54A05; 54A10; 54C08; 54D10

1. Introduction

Over the past several decades, a major trend in general topology has been to enrich topological spaces with additional set-theoretic or algebraic data. The ideal topological spaces $(X, \tau, \mathcal{I})$, whose roots go back to Kuratowski [1] and Vaidyanathaswamy [2], were brought into the mainstream by Janković and Hamlett [3]. In this framework, an ideal $\mathcal{I}$ (a hereditary, finitely additive family containing $\varnothing$) interacts with the topology through the local function $A^*(\mathcal{I}, \tau)$ and produces a finer topology τ^*. A *primal* $\mathcal{P}$ on (X, τ) is a dual notion to an ideal: $\mathcal{P}$ is a collection of subsets of X that is closed under supersets and satisfies $X \notin \mathcal{P}$; it was introduced by Acharjee, Özkoç, and Issaka [4] and has since generated substantial activity on operators and compatibility [5,6], separation axioms [7], and compactness [8]. Dual notions also followed: Cartan's filters [9], and Choquet's grills [10] revived by Roy and Mukherjee [11].

Alongside these developments, topology has been blended with fuzzy sets (Zadeh [12], Chang [13]), soft sets (Molodtsov [14], Shabir and Naz [15]), neutrosophic sets (Smaran-

Check for updates

Academic Editor: Manuel Sanchis

Received: 26 February 2026
Revised: 12 March 2026
Accepted: 16 March 2026
Published: 17 March 2026

https://doi.org/10.3390/math14061013

dache [16], Salama and Alblowi [17]), and rough sets (Pawlak [18]). The soft topology framework, in particular, has undergone rapid development in recent works, with significant advances in compactness [19], primal soft topology [20], and continuity properties [21]. In a parallel direction, Császár's generalized topology [22] and recent works on generalized open sets in various settings [23] have shown that weakening topological axioms can lead to rich and applicable theories. If one steps back and looks at the picture as a whole, the existing auxiliary structures can be grouped as follows: (i) subcollections of $\mathcal{P}(X)$ (ideals, filters, grills, primals); (ii) second topologies as in Kelly's bitopological spaces [24]; (iii) membership-grade functions (fuzzy, intuitionistic fuzzy, neutrosophic); (iv) parametric families (soft sets); and (v) equivalence relations (Pawlak rough sets).

The starting point of the present work is the observation that none of these structures captures the idea of assigning to each point a single fixed open neighborhood. We propose exactly this: a function $\mathfrak{a} : X \to \tau$ satisfying $x \in \mathfrak{a}(x)$ for all $x \in X$. We call $\mathfrak{a}$ a *scope function* and the triple $(X, \tau, \mathfrak{a})$ an *aura topological space*. The name is meant to suggest that each point x carries a "scope of influence" $\mathfrak{a}(x)$ around it. We stress that $\mathfrak{a}$ is not a subcollection of $\mathcal{P}(X)$, not a second topology, not a membership function, and not a relation; it is a point-to-open-set assignment.

We should note that neighborhood assignments $\phi : X \to \tau$ with $x \in \phi(x)$ already appear in the theory of D-spaces (van Douwen and Pfeffer [25]). There, however, such assignments serve as universally quantified variables—one says "for every neighborhood assignment $\phi\ldots$"—rather than as a fixed ingredient of the space. Our approach takes the opposite viewpoint: we fix $\mathfrak{a}$ once and for all and then derive operators, open-set classes, continuity notions, and separation axioms from it.

The need for such a structure becomes apparent when one considers phenomena that existing frameworks cannot naturally model. In ideal or primal topological spaces, the auxiliary structure is a single subcollection of the power set $\mathcal{P}(X)$; it does not distinguish individual points. By contrast, a scope function assigns different open neighborhoods to different points, making it possible to model *a variable-resolution observation*: a sensor network where each node has a different detection range, or a diagnostic system where each patient has a different similarity scope. Moreover, the failure of idempotency of the aura-closure operator $\mathrm{cl}_{\mathfrak{a}}$—a feature absent in all classical auxiliary structures—gives rise to a natural transfinite iteration process that produces the equalities and inclusions $\tau_{\mathfrak{a}}^{\infty} = \tau_{\mathfrak{a}} \subseteq \tau$, capturing different levels of "reachability" from the original topology. The comparison table in Section 7 provides a systematic comparison with ideals, filters, grills, and primals.

The paper is structured as follows. Section 2 recalls the necessary preliminaries. Section 3 introduces aura topological spaces and establishes the basic properties of $\mathrm{cl}_{\mathfrak{a}}$ and $\mathrm{int}_{\mathfrak{a}}$; in particular, $\mathrm{cl}_{\mathfrak{a}}$ is shown to be an additive Čech closure operator whose iteration $\mathrm{cl}_{\mathfrak{a}}^{\infty}$ is a Kuratowski closure, and the topology $\tau_{\mathfrak{a}}$ of $\mathfrak{a}$-open sets is proved to satisfy $\tau_{\mathfrak{a}}^{\infty} = \tau_{\mathfrak{a}} \subseteq \tau$. In Section 4, we define five generalized open-set classes and construct a complete implication diagram, using counterexamples on both finite spaces and the real line. Section 5 treats $\mathfrak{a}$-continuity, its decompositions, and $\mathfrak{a}$-convergence of sequences. Section 6 introduces $\mathfrak{a}$-T_0, $\mathfrak{a}$-T_1, and $\mathfrak{a}$-T_2 separation axioms, and establishes the equivalence of $\mathfrak{a}$-T_1 and $\mathfrak{a}$-T_2. Section 7 provides a detailed comparison of the aura framework with existing auxiliary structures. Section 8 gathers concluding remarks and open problems.

2. Preliminaries

Throughout this paper, (X, τ) denotes a topological space. For a subset A of X, the closure of A in (X, τ) is denoted by $\mathrm{cl}(A)$ and the interior by $\mathrm{int}(A)$. The complement of A

is denoted by $A^c = X \setminus A$. The power set of X is denoted by $\mathcal{P}(X)$. For $x \in X$, we denote by $\tau(x) = \{U \in \tau : x \in U\}$ the collection of all open neighborhoods of x.

Definition 1 ([26])**.** *A subset A of a topological space (X, τ) is called* semi-open *if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$. The complement of a semi-open set is called* semi-closed. *The collection of all semi-open sets is denoted by* $\mathrm{SO}(X, \tau)$.

Definition 2 ([27])**.** *A subset A of a topological space (X, τ) is called* pre-open *if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. The complement of a pre-open set is called* pre-closed. *The collection of all pre-open sets is denoted by* $\mathrm{PO}(X, \tau)$.

Definition 3 ([28])**.** *A subset A of a topological space (X, τ) is called* α-open *if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. The collection of all α-open sets is denoted by $\alpha O(X, \tau)$.*

Definition 4 ([29])**.** *A subset A of a topological space (X, τ) is called* β-open *(or* semi-pre-open*) if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. The collection of all β-open sets is denoted by $\beta O(X, \tau)$.*

Definition 5 ([30])**.** *A function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a* Čech closure operator *if it satisfies the following conditions for all $A, B \subseteq X$:*

(C1) $c(\varnothing) = \varnothing$;
(C2) $A \subseteq c(A)$ (extensivity);
(C3) $A \subseteq B \Rightarrow c(A) \subseteq c(B)$ (monotonicity).

If c additionally satisfies

(C4) $c(A \cup B) = c(A) \cup c(B)$ (finite additivity),

then c is called an additive Čech closure operator. *If c further satisfies*

(C5) $c(c(A)) = c(A)$ (idempotency),

then c is a Kuratowski closure operator *and generates a topology on X.*

3. Aura Topological Spaces

In this section, we introduce the central concept of this paper and investigate the fundamental properties of the associated operators.

Definition 6. *Let (X, τ) be a topological space. A function $\mathfrak{a} : X \to \tau$ is called a* scope function *(or* aura function*) on (X, τ) if it satisfies*

$$x \in \mathfrak{a}(x) \quad \text{for every } x \in X. \tag{1}$$

The triple $(X, \tau, \mathfrak{a})$ is called an aura topological space *(briefly, an $\mathfrak{a}$-space).*

The scope function $\mathfrak{a}$ selects, for each point x, a particular open neighborhood $\mathfrak{a}(x) \in \tau(x)$. One may think of $\mathfrak{a}(x)$ as the "scope" of x—the part of the space that x can directly interact with. Different choices of $\mathfrak{a}$ on the same topological space (X, τ) yield different $\mathfrak{a}$-spaces with potentially different properties.

Example 1. *Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Define $\mathfrak{a} : X \to \tau$ by $\mathfrak{a}(a) = \{a\}$, $\mathfrak{a}(b) = \{a, b\}$, $\mathfrak{a}(c) = \{a, b, c\}$, $\mathfrak{a}(d) = X$. Then, $(X, \tau, \mathfrak{a})$ is an $\mathfrak{a}$-space since $x \in \mathfrak{a}(x)$ for each $x \in X$.*

Example 2. *Let $(\mathbb{R}, \tau_u)$ be the real line with the usual topology. For a fixed $\varepsilon > 0$, define $\mathfrak{a}_\varepsilon : \mathbb{R} \to \tau_u$ by $\mathfrak{a}_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ for every $x \in \mathbb{R}$. Then, $(\mathbb{R}, \tau_u, \mathfrak{a}_\varepsilon)$ is an $\mathfrak{a}$-space. This can be interpreted as assigning each point a fixed "resolution window" of radius ε.*

Example 3. *Let* $(\mathbb{R}, \tau_u)$ *be as above. Define* $\mathfrak{a} : \mathbb{R} \to \tau_u$ *by*

$$\mathfrak{a}(x) = \left(x - \tfrac{1}{1+x^2},\ x + \tfrac{1}{1+x^2}\right)$$

for every $x \in \mathbb{R}$*. Then,* $(\mathbb{R}, \tau_u, \mathfrak{a})$ *is an* $\mathfrak{a}$*-space where the scope narrows as* $|x| \to \infty$*, modeling a variable-resolution observation.*

3.1. The Aura-Closure Operator

Definition 7. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. For* $A \subseteq X$*, the* aura-closure *of* A *is defined by*

$$\mathrm{cl}_{\mathfrak{a}}(A) = \{x \in X : \mathfrak{a}(x) \cap A \neq \varnothing\}. \tag{2}$$

The aura-closure of A *consists of all points whose aura intersects* A*; equivalently,* $x \in \mathrm{cl}_{\mathfrak{a}}(A)$ *if and only if* A *meets* $\mathfrak{a}(x)$*. These points may be called the* $\mathfrak{a}$-adherence points *of* A*.*

Theorem 1. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. The operator* $\mathrm{cl}_{\mathfrak{a}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *satisfies the following properties for all* $A, B \subseteq X$*:*

(a) $\mathrm{cl}_{\mathfrak{a}}(\varnothing) = \varnothing$.
(b) $A \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$.
(c) $A \subseteq B \Rightarrow \mathrm{cl}_{\mathfrak{a}}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(B)$.
(d) $\mathrm{cl}_{\mathfrak{a}}(A \cup B) = \mathrm{cl}_{\mathfrak{a}}(A) \cup \mathrm{cl}_{\mathfrak{a}}(B)$.
(e) $\mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$.

Hence, $\mathrm{cl}_{\mathfrak{a}}$ *is an additive Čech closure operator satisfying the first four Kuratowski axioms.*

Proof. (a) Since $\mathfrak{a}(x) \cap \varnothing = \varnothing$ for every $x \in X$, we have $\mathrm{cl}_{\mathfrak{a}}(\varnothing) = \varnothing$.

(b) Let $x \in A$. By (1), $x \in \mathfrak{a}(x)$, so $x \in \mathfrak{a}(x) \cap A \neq \varnothing$. Thus $x \in \mathrm{cl}_{\mathfrak{a}}(A)$.

(c) Let $A \subseteq B$ and $x \in \mathrm{cl}_{\mathfrak{a}}(A)$. Then $\mathfrak{a}(x) \cap A \neq \varnothing$, so there exists $y \in \mathfrak{a}(x) \cap A \subseteq \mathfrak{a}(x) \cap B$. Hence $x \in \mathrm{cl}_{\mathfrak{a}}(B)$.

(d) For any $x \in X$,

$$\begin{aligned} x \in \mathrm{cl}_{\mathfrak{a}}(A \cup B) &\iff \mathfrak{a}(x) \cap (A \cup B) \neq \varnothing \\ &\iff (\mathfrak{a}(x) \cap A) \cup (\mathfrak{a}(x) \cap B) \neq \varnothing \\ &\iff x \in \mathrm{cl}_{\mathfrak{a}}(A) \text{ or } x \in \mathrm{cl}_{\mathfrak{a}}(B) \\ &\iff x \in \mathrm{cl}_{\mathfrak{a}}(A) \cup \mathrm{cl}_{\mathfrak{a}}(B). \end{aligned}$$

(e) Let $x \in \mathrm{cl}(A)$. Then, every open neighborhood of x intersects A. Since $\mathfrak{a}(x)$ is an open neighborhood of x, we have $\mathfrak{a}(x) \cap A \neq \varnothing$, so $x \in \mathrm{cl}_{\mathfrak{a}}(A)$. □

Remark 1. *The operator* $\mathrm{cl}_{\mathfrak{a}}$ *need not be idempotent, and this is its most fundamental distinction from classical closure operators. Precisely because* $\mathrm{cl}_{\mathfrak{a}}$ *fails idempotency, it cannot be a Kuratowski closure operator in general, and the transfinite iteration introduced in Section 3.4 becomes necessary.*

Theorem 2. *The operator* $\mathrm{cl}_{\mathfrak{a}}$ *is not idempotent in general; that is, there exists an* $\mathfrak{a}$*-space* $(X, \tau, \mathfrak{a})$ *and a subset* $A \subseteq X$ *such that* $\mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}_{\mathfrak{a}}(A)) \neq \mathrm{cl}_{\mathfrak{a}}(A)$*.*

Proof. Let $X = \{a, b, c\}$, $\tau = \mathcal{P}(X)$ (discrete topology), and define $\mathfrak{a}(a) = \{a, b\}$, $\mathfrak{a}(b) = \{b, c\}$, $\mathfrak{a}(c) = \{c\}$. Let $A = \{c\}$. Then, $\mathrm{cl}_{\mathfrak{a}}(A) = \{b, c\}$, since $\mathfrak{a}(b) \cap \{c\} = \{c\} \neq \varnothing$ and $\mathfrak{a}(c) \cap \{c\} = \{c\} \neq \varnothing$, but $\mathfrak{a}(a) \cap \{c\} = \varnothing$. Now $\mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}_{\mathfrak{a}}(A)) = \mathrm{cl}_{\mathfrak{a}}(\{b, c\}) = X$, since $\mathfrak{a}(a) \cap \{b, c\} = \{b\} \neq \varnothing$. Therefore, $\mathrm{cl}_{\mathfrak{a}}(A) = \{b, c\} \neq X = \mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}_{\mathfrak{a}}(A))$. □

Theorem 3. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space and* $A \subseteq X$*. Then,*

(a) $\mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$.
(b) *The inclusion can be strict.*
(c) $\mathrm{cl}_{\mathfrak{a}}(A) \subseteq \bigcup_{x\in A} \mathfrak{a}(x)$ *need not hold in general.*

Proof. Part (a) is Theorem 1(e). For part (b), consider $X = \{a, b, c\}$, $\tau = \mathcal{P}(X)$, $\mathfrak{a}(a) = \{a, b\}$, $\mathfrak{a}(b) = \{b\}$, $\mathfrak{a}(c) = \{c\}$, and $A = \{b\}$. Then, $\mathrm{cl}(A) = \{b\}$ (discrete topology) but $\mathrm{cl}_{\mathfrak{a}}(A) = \{a, b\}$, since $\mathfrak{a}(a) \cap \{b\} \neq \varnothing$. The inclusion is strict. For part (c), with the same space, $\bigcup_{x\in A} \mathfrak{a}(x) = \mathfrak{a}(b) = \{b\}$, but $\mathrm{cl}_{\mathfrak{a}}(A) = \{a, b\} \not\subseteq \{b\}$. □

3.2. The Aura-Interior Operator

Definition 8. *Let $(X, \tau, \mathfrak{a})$ be an $\mathfrak{a}$-space. For $A \subseteq X$, the* aura-interior *of A is defined by*

$$\mathrm{int}_{\mathfrak{a}}(A) = \{x \in A : \mathfrak{a}(x) \subseteq A\}. \tag{3}$$

Theorem 4. *Let $(X, \tau, \mathfrak{a})$ be an $\mathfrak{a}$-space. The operator $\mathrm{int}_{\mathfrak{a}} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the following for all $A, B \subseteq X$:*

(a) $\mathrm{int}_{\mathfrak{a}}(X) = X$.
(b) $\mathrm{int}_{\mathfrak{a}}(\varnothing) = \varnothing$.
(c) $\mathrm{int}_{\mathfrak{a}}(A) \subseteq A$.
(d) $A \subseteq B \Rightarrow \mathrm{int}_{\mathfrak{a}}(A) \subseteq \mathrm{int}_{\mathfrak{a}}(B)$.
(e) $\mathrm{int}_{\mathfrak{a}}(A \cap B) = \mathrm{int}_{\mathfrak{a}}(A) \cap \mathrm{int}_{\mathfrak{a}}(B)$.
(f) $\mathrm{int}_{\mathfrak{a}}(A) \subseteq \mathrm{int}(A)$.
(g) $\mathrm{int}_{\mathfrak{a}}(A) = A \setminus \mathrm{cl}_{\mathfrak{a}}(A^c)$.

Proof. (a) For every $x \in X$, $\mathfrak{a}(x) \subseteq X$, so $x \in \mathrm{int}_{\mathfrak{a}}(X)$.

(b) There is no $x \in \varnothing$, so $\mathrm{int}_{\mathfrak{a}}(\varnothing) = \varnothing$.

(c) By definition, $\mathrm{int}_{\mathfrak{a}}(A) = \{x \in A : \mathfrak{a}(x) \subseteq A\} \subseteq A$.

(d) Let $x \in \mathrm{int}_{\mathfrak{a}}(A)$. Then, $x \in A \subseteq B$ and $\mathfrak{a}(x) \subseteq A \subseteq B$, so $x \in \mathrm{int}_{\mathfrak{a}}(B)$.

(e) Let $x \in \mathrm{int}_{\mathfrak{a}}(A \cap B)$. Then $x \in A \cap B$ and $\mathfrak{a}(x) \subseteq A \cap B$, so $x \in \mathrm{int}_{\mathfrak{a}}(A) \cap \mathrm{int}_{\mathfrak{a}}(B)$. Conversely, if $x \in \mathrm{int}_{\mathfrak{a}}(A) \cap \mathrm{int}_{\mathfrak{a}}(B)$, then $x \in A \cap B$ and $\mathfrak{a}(x) \subseteq A \cap B$, giving $x \in \mathrm{int}_{\mathfrak{a}}(A \cap B)$.

(f) Let $x \in \mathrm{int}_{\mathfrak{a}}(A)$. Then, $x \in A$ and $\mathfrak{a}(x) \subseteq A$. Since $\mathfrak{a}(x)$ is an open set containing x and contained in A, we have $x \in \mathrm{int}(A)$.

(g) $x \in \mathrm{int}_{\mathfrak{a}}(A) \Leftrightarrow x \in A$ and $\mathfrak{a}(x) \subseteq A \Leftrightarrow x \in A$ and $\mathfrak{a}(x) \cap A^c = \varnothing \Leftrightarrow x \in A$ and $x \notin \mathrm{cl}_{\mathfrak{a}}(A^c) \Leftrightarrow x \in A \setminus \mathrm{cl}_{\mathfrak{a}}(A^c)$. □

Remark 2. *The inclusion $\mathrm{int}_{\mathfrak{a}}(A) \subseteq \mathrm{int}(A)$ in Theorem 4(f) can be strict: in Example 1, $\mathrm{int}(\{a, b\}) = \{a, b\}$ but $\mathrm{int}_{\mathfrak{a}}(\{a, b\}) = \{a\}$ since $\mathfrak{a}(b) = \{a, b\} \subseteq \{a, b\}$ (so $b \in \mathrm{int}_{\mathfrak{a}}$). Actually, $\mathfrak{a}(b) = \{a, b\} \subseteq \{a, b\}$, so both a and b are in $\mathrm{int}_{\mathfrak{a}}(\{a, b\})$. For strict inclusion, take $X = \{a, b\}$, $\tau = \mathcal{P}(X)$, $\mathfrak{a}(a) = \{a, b\}$, $\mathfrak{a}(b) = \{b\}$, and $A = \{a\}$. Then, $\mathrm{int}(\{a\}) = \{a\}$ but $\mathrm{int}_{\mathfrak{a}}(\{a\}) = \varnothing$ since $\mathfrak{a}(a) = \{a, b\} \not\subseteq \{a\}$.*

3.3. The Aura Topology

Definition 9. *Let $(X, \tau, \mathfrak{a})$ be an $\mathfrak{a}$-space. A subset A of X is called* $\mathfrak{a}$-open *if for every $x \in A$, $\mathfrak{a}(x) \subseteq A$. Equivalently, A is $\mathfrak{a}$-open if and only if $\mathrm{int}_{\mathfrak{a}}(A) = A$. The collection of all $\mathfrak{a}$-open sets is denoted by $\tau_{\mathfrak{a}}$.*

Theorem 5. *Let $(X, \tau, \mathfrak{a})$ be an $\mathfrak{a}$-space. Then,*

(a) *$\tau_{\mathfrak{a}}$ is a topology on X.*
(b) *$\tau_{\mathfrak{a}} \subseteq \tau$.*

Proof. (a) Topology axioms: $\varnothing \in \tau_{\mathfrak{a}}$ vacuously; $X \in \tau_{\mathfrak{a}}$ since $\mathfrak{a}(x) \subseteq X$ for all x; finite intersections: if $A, B \in \tau_{\mathfrak{a}}$ and $x \in A \cap B$, then $\mathfrak{a}(x) \subseteq A$ and $\mathfrak{a}(x) \subseteq B$, so $\mathfrak{a}(x) \subseteq A \cap B$; arbitrary unions: if $x \in \bigcup_i A_i$ with $A_j \in \tau_{\mathfrak{a}}$ and $x \in A_j$, then $\mathfrak{a}(x) \subseteq A_j \subseteq \bigcup_i A_i$.

(b) Let $A \in \tau_{\mathfrak{a}}$. For every $x \in A$, $\mathfrak{a}(x) \subseteq A$ and $\mathfrak{a}(x) \in \tau$ with $x \in \mathfrak{a}(x)$. Thus, $A = \bigcup_{x \in A} \mathfrak{a}(x) \in \tau$. □

Example 4. *In Example 1, the* $\mathfrak{a}$*-open sets are* $\tau_{\mathfrak{a}} = \{\varnothing, \{a\}, \{a,b\}, \{a,b,c\}, X\}$*. Note that* $\{b\} \in \tau$ *but* $\{b\} \notin \tau_{\mathfrak{a}}$*, confirming* $\tau_{\mathfrak{a}} \subsetneq \tau$ *in general.*

Proposition 1. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. Every* $\mathfrak{a}$*-open set is a union of members of* $\mathcal{B}_{\mathfrak{a}} = \{\mathfrak{a}(x) : x \in X\}$*; that is, if* $A \in \tau_{\mathfrak{a}}$*, then* $A = \bigcup_{x \in A} \mathfrak{a}(x)$*. In general, however,* $\mathcal{B}_{\mathfrak{a}}$ *need not be a base for* $\tau_{\mathfrak{a}}$*.*

Definition 10. *An* $\mathfrak{a}$*-space* $(X, \tau, \mathfrak{a})$ *is called* transitive *if for every* $x \in X$ *and every* $y \in \mathfrak{a}(x)$*,* $\mathfrak{a}(y) \subseteq \mathfrak{a}(x)$*.*

Proposition 2. *If* $(X, \tau, \mathfrak{a})$ *is a transitive* $\mathfrak{a}$*-space, then* $\mathcal{B}_{\mathfrak{a}} = \{\mathfrak{a}(x) : x \in X\}$ *is a base for* $\tau_{\mathfrak{a}}$*, and the aura-closure* $\mathrm{cl}_{\mathfrak{a}}$ *is idempotent.*

3.4. Iterative Aura Closure

Since $\mathrm{cl}_{\mathfrak{a}}$ is not idempotent in general, we define the iterative closure.

Definition 11. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space and* $A \subseteq X$*. Define*

$$\mathrm{cl}^0_{\mathfrak{a}}(A) = A, \tag{4}$$

$$\mathrm{cl}^{n+1}_{\mathfrak{a}}(A) = \mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}^n_{\mathfrak{a}}(A)) \quad (n \geq 0), \tag{5}$$

$$\mathrm{cl}^{\infty}_{\mathfrak{a}}(A) = \bigcup_{n=0}^{\infty} \mathrm{cl}^n_{\mathfrak{a}}(A). \tag{6}$$

Theorem 6. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space and* $A \subseteq X$*. Then,*

(a) $A \subseteq \mathrm{cl}_{\mathfrak{a}}(A) \subseteq \mathrm{cl}^2_{\mathfrak{a}}(A) \subseteq \cdots \subseteq \mathrm{cl}^{\infty}_{\mathfrak{a}}(A)$.

(b) $\mathrm{cl}^{\infty}_{\mathfrak{a}}$ *is a Kuratowski closure operator.*

(c) *The topology generated by* $\mathrm{cl}^{\infty}_{\mathfrak{a}}$ *satisfies* $\tau^{\infty}_{\mathfrak{a}} = \tau_{\mathfrak{a}} \subseteq \tau$*.*

Proof. (a) By extensivity of $\mathrm{cl}_{\mathfrak{a}}$, $\mathrm{cl}^n_{\mathfrak{a}}(A) \subseteq \mathrm{cl}^{n+1}_{\mathfrak{a}}(A)$ for every $n \geq 0$.

(b) We verify the Kuratowski axioms for $\mathrm{cl}^{\infty}_{\mathfrak{a}}$:

- $\mathrm{cl}^{\infty}_{\mathfrak{a}}(\varnothing) = \bigcup_n \mathrm{cl}^n_{\mathfrak{a}}(\varnothing) = \varnothing$.
- $A \subseteq \mathrm{cl}^{\infty}_{\mathfrak{a}}(A)$: since $A = \mathrm{cl}^0_{\mathfrak{a}}(A)$.
- Additivity: $\mathrm{cl}^{\infty}_{\mathfrak{a}}(A \cup B) = \bigcup_n \mathrm{cl}^n_{\mathfrak{a}}(A \cup B) = \bigcup_n (\mathrm{cl}^n_{\mathfrak{a}}(A) \cup \mathrm{cl}^n_{\mathfrak{a}}(B)) = \mathrm{cl}^{\infty}_{\mathfrak{a}}(A) \cup \mathrm{cl}^{\infty}_{\mathfrak{a}}(B)$.
- Idempotency: Let $x \in \mathrm{cl}^{\infty}_{\mathfrak{a}}(\mathrm{cl}^{\infty}_{\mathfrak{a}}(A))$. Then, $x \in \mathrm{cl}^m_{\mathfrak{a}}(\mathrm{cl}^{\infty}_{\mathfrak{a}}(A))$ for some m. For $m = 0$, $x \in \mathrm{cl}^{\infty}_{\mathfrak{a}}(A)$. For $m = 1$, $x \in \mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}^{\infty}_{\mathfrak{a}}(A))$ means $\mathfrak{a}(x) \cap \mathrm{cl}^{\infty}_{\mathfrak{a}}(A) \neq \varnothing$, so there exist $y \in \mathfrak{a}(x)$ and k with $y \in \mathrm{cl}^k_{\mathfrak{a}}(A)$. Then, $x \in \mathrm{cl}_{\mathfrak{a}}(\mathrm{cl}^k_{\mathfrak{a}}(A)) = \mathrm{cl}^{k+1}_{\mathfrak{a}}(A) \subseteq \mathrm{cl}^{\infty}_{\mathfrak{a}}(A)$. The general case follows by induction on m.

(c) We show $\tau^{\infty}_{\mathfrak{a}} = \tau_{\mathfrak{a}}$. A set $F \subseteq X$ is $\tau^{\infty}_{\mathfrak{a}}$-closed if and only if $\mathrm{cl}^{\infty}_{\mathfrak{a}}(F) = F$. Since $F = \mathrm{cl}^0_{\mathfrak{a}}(F) \subseteq \mathrm{cl}^1_{\mathfrak{a}}(F) \subseteq \cdots$, the equality $\mathrm{cl}^{\infty}_{\mathfrak{a}}(F) = \bigcup_n \mathrm{cl}^n_{\mathfrak{a}}(F) = F$ holds if and only if $\mathrm{cl}^1_{\mathfrak{a}}(F) = \mathrm{cl}_{\mathfrak{a}}(F) \subseteq F$. Combined with $F \subseteq \mathrm{cl}_{\mathfrak{a}}(F)$ (extensivity), this is equivalent to $\mathrm{cl}_{\mathfrak{a}}(F) = F$. Now, F is $\tau_{\mathfrak{a}}$-closed if and only if $X \setminus F \in \tau_{\mathfrak{a}}$; that is, for every $x \in X \setminus F$, $\mathfrak{a}(x) \subseteq X \setminus F$, which means $\mathfrak{a}(x) \cap F = \varnothing$ for every $x \notin F$, i.e., $\mathrm{cl}_{\mathfrak{a}}(F) \subseteq F$, equivalently $\mathrm{cl}_{\mathfrak{a}}(F) = F$. Therefore $\tau^{\infty}_{\mathfrak{a}}$-closed sets coincide with $\tau_{\mathfrak{a}}$-closed sets, so $\tau^{\infty}_{\mathfrak{a}} = \tau_{\mathfrak{a}}$. The inclusion $\tau_{\mathfrak{a}} \subseteq \tau$ follows from Theorem 5(b). □

Remark 3. *Theorem 6(c) reveals a notable rigidity of the aura framework: the Kuratowski closure* $\mathrm{cl}_{\mathfrak{a}}^{\infty}$ *generates exactly the same topology as the non-idempotent Čech closure* $\mathrm{cl}_{\mathfrak{a}}$. *This means the transfinite iteration process does not produce a genuinely new topology; instead, the topology chain collapses to* $\tau_{\mathfrak{a}}^{\infty} = \tau_{\mathfrak{a}} \subseteq \tau$. *The interesting topological data is therefore entirely captured by* $\tau_{\mathfrak{a}}$ *and* τ.

Note 1. If X is finite, then the chain $\mathrm{cl}_{\mathfrak{a}}^{0}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}^{1}(A) \subseteq \cdots$ stabilizes in at most $|X|$ steps, so $\mathrm{cl}_{\mathfrak{a}}^{\infty}(A) = \mathrm{cl}_{\mathfrak{a}}^{|X|}(A)$.

3.5. Special Types of Aura Functions

Definition 12. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. We say that* $\mathfrak{a}$ *is:*

(a) trivial *if* $\mathfrak{a}(x) = X$ *for every* $x \in X$;
(b) discrete *if* $\mathfrak{a}(x) = \{x\}$ *for every* $x \in X$ *(possible only if* τ *is the discrete topology);*
(c) monotone *(or* transitive*) if* $x \in \mathfrak{a}(y)$ *implies* $\mathfrak{a}(x) \subseteq \mathfrak{a}(y)$;
(d) symmetric *if* $y \in \mathfrak{a}(x)$ *implies* $x \in \mathfrak{a}(y)$.

Proposition 3. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space.*

(a) *If* $\mathfrak{a}$ *is trivial, then* $\mathrm{cl}_{\mathfrak{a}}(A) = X$ *for every nonempty* A, *and* $\tau_{\mathfrak{a}} = \{\varnothing, X\}$.
(b) *If* $\mathfrak{a}$ *is discrete, then* $\mathrm{cl}_{\mathfrak{a}}(A) = A$ *for every* A, *and* $\tau_{\mathfrak{a}} = \mathcal{P}(X)$.
(c) *If* $\mathfrak{a}$ *is symmetric, then* $x \in \mathrm{cl}_{\mathfrak{a}}(\{y\})$ *if and only if* $y \in \mathrm{cl}_{\mathfrak{a}}(\{x\})$.

4. Generalized Open Sets in Aura Spaces

In this section, we introduce five new classes of generalized open sets by combining the aura-closure operator $\mathrm{cl}_{\mathfrak{a}}$ with the classical interior and closure operators. These generalize the classical notions of semi-open [26], pre-open [27], α-open [28], β-open [29], and b-open [31] sets to the aura setting.

Definition 13. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. A subset* A *of* X *is called:*

(a) $\mathfrak{a}$-semi-open *if* $A \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))$;
(b) $\mathfrak{a}$-pre-open *if* $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$;
(c) $\mathfrak{a}$-α-open *if* $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)))$;
(d) $\mathfrak{a}$-β-open *if* $A \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)))$;
(e) $\mathfrak{a}$-b-open *if* $A \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$.

The collection of all $\mathfrak{a}$*-semi-open (resp.* $\mathfrak{a}$*-pre-open,* $\mathfrak{a}$*-*α*-open,* $\mathfrak{a}$*-*β*-open,* $\mathfrak{a}$*-*b*-open) sets is denoted by* $a\mathrm{SO}(X)$ *(resp.* $a\mathrm{PO}(X)$, $a\alpha\mathrm{O}(X)$, $a\beta\mathrm{O}(X)$, $ab\mathrm{O}(X)$*).*

Theorem 7 (Implication hierarchy)**.** *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. The following inclusions hold:*

$$\tau \subseteq a\alpha\mathrm{O}(X) \quad \Rightarrow \quad \begin{cases} a\mathrm{SO}(X) \\ a\mathrm{PO}(X) \end{cases} \subseteq \; ab\mathrm{O}(X) \; \subseteq \; a\beta\mathrm{O}(X),$$

where $A \to B$ *means every* A*-type set is also a* B*-type set. Moreover, each implication is strict in general.*

Proof. $\tau \subseteq a\alpha\mathrm{O}(X)$: Let $A \in \tau$. Then, $\mathrm{int}(A) = A$, so $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) = \mathrm{cl}_{\mathfrak{a}}(A) \supseteq \mathrm{cl}(A) \supseteq A$. Hence, $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))) \supseteq A$.

$a\alpha\mathrm{O}(X) \subseteq a\mathrm{SO}(X)$: If $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)))$, then $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))) \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))$.

$a\alpha\mathrm{O}(X) \subseteq a\mathrm{PO}(X)$: Since $\mathrm{int}(A) \subseteq A$ gives $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$, we get $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))) \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$.

$a\mathrm{SO}(X) \subseteq ab\mathrm{O}(X)$: $A \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$.

$a\mathrm{PO}(X) \subseteq ab\mathrm{O}(X)$: Similarly.

$ab\mathrm{O}(X) \subseteq a\beta\mathrm{O}(X)$: Note $\mathrm{int}(A) \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$, so $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)))$. Also, $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)) \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)))$ by extensivity. Hence, $A \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)))$. □

Theorem 8. *Let $(X, \tau, \mathfrak{a})$ be an $\mathfrak{a}$-space. Since $\mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$ for all $A \subseteq X$, we have:*

(a) $\mathrm{SO}(X, \tau) \subseteq a\mathrm{SO}(X)$;
(b) $\mathrm{PO}(X, \tau) \subseteq a\mathrm{PO}(X)$;
(c) $\alpha O(X, \tau) \subseteq a\alpha\mathrm{O}(X)$;
(d) $\beta O(X, \tau) \subseteq a\beta\mathrm{O}(X)$.

Each inclusion can be strict.

Proof. Since $\mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$, we have $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) \supseteq \mathrm{cl}(\mathrm{int}(A))$. If $A \in \mathrm{SO}(X, \tau)$, then $A \subseteq \mathrm{cl}(\mathrm{int}(A)) \subseteq \mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))$, so $A \in a\mathrm{SO}(X)$. The other parts follow similarly. □

We now provide counterexamples demonstrating that each inclusion in the hierarchy is strict.

Example 5. *[Classical semi-open $\subsetneq$ $\mathfrak{a}$-semi-open] Consider $(\mathbb{R}, \tau_u, \mathfrak{a}_2)$ with $\mathfrak{a}_2(x) = (x-2, x+2)$. Let $A = \{0\} \cup (1,2)$. Then, $\mathrm{int}(A) = (1,2)$ and $\mathrm{cl}((1,2)) = [1,2]$. Since $0 \notin [1,2]$, the set A is not classically semi-open. On the other hand, $\mathrm{cl}_{\mathfrak{a}}((1,2)) = \{x \in \mathbb{R} : (x-2, x+2) \cap (1,2) \neq \varnothing\} = (-1,4)$. Since $A \subseteq (-1,4)$, the set A is $\mathfrak{a}_2$-semi-open.*

Example 6 ($\mathfrak{a}$-semi-open $\not\Rightarrow$ $\mathfrak{a}$-pre-open). *Let $X = \{a,b,c,d\}$, $\tau = \{\varnothing, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, X\}$, and $\mathfrak{a}(a) = \{a\}$, $\mathfrak{a}(b) = \{a,b,c\}$, $\mathfrak{a}(c) = \{c\}$, $\mathfrak{a}(d) = X$. Let $A = \{a,b\}$. Then, $\mathrm{int}(A) = \{a\}$ and $\mathrm{cl}_{\mathfrak{a}}(\{a\}) = \{a,b,d\}$. Since $A \subseteq \{a,b,d\}$, the set A is $\mathfrak{a}$-semi-open. However, $\mathrm{cl}_{\mathfrak{a}}(A) = \{a,b,d\}$ and $\mathrm{int}(\{a,b,d\}) = \{a\}$. Since $b \in A$ but $b \notin \{a\}$, we have $A \nsubseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A))$, so A is not $\mathfrak{a}$-pre-open.*

Example 7 ($\mathfrak{a}$-pre-open $\not\Rightarrow$ $\mathfrak{a}$-semi-open). *Let $X = \{a,b,c,d\}$, $\tau = \{\varnothing, \{a,b\}, \{c,d\}, X\}$, and $\mathfrak{a}(a) = \{a,b\}$, $\mathfrak{a}(b) = \{a,b\}$, $\mathfrak{a}(c) = \{c,d\}$, $\mathfrak{a}(d) = X$. Let $A = \{a,c\}$. Since A contains no nonempty open set, $\mathrm{int}(A) = \varnothing$, hence $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A)) = \varnothing$ and A is not $\mathfrak{a}$-semi-open. On the other hand, $\mathrm{cl}_{\mathfrak{a}}(A) = X$ (every aura meets $\{a,c\}$), so $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(A)) = X \supseteq A$, and A is $\mathfrak{a}$-pre-open.*

Example 8 ($\mathfrak{a}$-semi-open $\not\Rightarrow$ $\mathfrak{a}$-α-open). *In Example 6, $A = \{a,b\}$ is $\mathfrak{a}$-semi-open. However, $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(A))) = \mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\{a\})) = \mathrm{int}(\{a,b,d\}) = \{a\}$, and $A = \{a,b\} \nsubseteq \{a\}$, so A is not $\mathfrak{a}$-α-open.*

Example 9 (Real-line illustrations). *Consider $(\mathbb{R}, \tau_u, \mathfrak{a}_2)$ with $\mathfrak{a}_2(x) = (x-2, x+2)$.*

(i) *$A = \{0\} \cup (1,2)$ is $\mathfrak{a}_2$-semi-open but not classically semi-open (Example 5).*
(ii) *$B = \{0\} \cup (3,4)$: $\mathrm{cl}_{\mathfrak{a}}(B) = (-2,2) \cup (1,6) = (-2,6)$ and $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(B)) = (-2,6) \supseteq B$, so B is $\mathfrak{a}_2$-pre-open. But $\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(B)) = \mathrm{cl}_{\mathfrak{a}}((3,4)) = (1,6)$ and $0 < 1$, so $B \nsubseteq (1,6)$; hence, B is not $\mathfrak{a}_2$-semi-open.*
(iii) *$C = [0,1]$: $\mathrm{int}(C) = (0,1)$ and $\mathrm{cl}_{\mathfrak{a}}((0,1)) = (-2,3) \supseteq C$. Moreover, $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}}(\mathrm{int}(C))) = (-2,3) \supseteq C$, so C is $\mathfrak{a}_2$-α-open.*

5. Aura-Continuity, Decomposition, and Convergence

The classical decomposition of continuity via semi-continuity and pre-continuity has been a central theme since the works of Levine [26], Mashhour et al. [27,32], and Reilly–Vamanamurthy [33]. In this section, we develop the aura-versions of these continuity notions, establish analogous decomposition results, and introduce the notion of $\mathfrak{a}$-convergence of sequences.

Definition 14. *Let* $(X, \tau_1, \mathfrak{a}_1)$ *and* $(Y, \tau_2, \mathfrak{a}_2)$ *be* $\mathfrak{a}$*-spaces. A function* $f : X \to Y$ *is called:*

(a) $\mathfrak{a}$-continuous *if* $f^{-1}(V) \in \tau_{\mathfrak{a}_1}$ *for every* $V \in \tau_{\mathfrak{a}_2}$*;*
(b) $\mathfrak{a}$-semi-continuous *if* $f^{-1}(V) \in a_1\mathrm{SO}(X)$ *for every* $V \in \tau_2$*;*
(c) $\mathfrak{a}$-pre-continuous *if* $f^{-1}(V) \in a_1\mathrm{PO}(X)$ *for every* $V \in \tau_2$*;*
(d) $\mathfrak{a}$-α-continuous *if* $f^{-1}(V) \in a_1\alpha\mathrm{O}(X)$ *for every* $V \in \tau_2$*;*
(e) $\mathfrak{a}$-β-continuous *if* $f^{-1}(V) \in a_1\beta\mathrm{O}(X)$ *for every* $V \in \tau_2$*.*

Theorem 9. *The following implications hold for any function* $f : (X, \tau_1, \mathfrak{a}_1) \to (Y, \tau_2, \mathfrak{a}_2)$*:*

$$\textit{continuous} \Rightarrow \mathfrak{a}\text{-}\alpha\textit{-continuous} \Rightarrow \begin{cases} \mathfrak{a}\textit{-semi-continuous} \\ \mathfrak{a}\textit{-pre-continuous} \end{cases} \Rightarrow \mathfrak{a}\text{-}\beta\textit{-continuous}.$$

Proof. These follow directly from Theorem 7. ☐

Theorem 10 (Composition)**.** *Let* $(X, \tau_1, \mathfrak{a}_1)$*,* $(Y, \tau_2, \mathfrak{a}_2)$*, and* $(Z, \tau_3, \mathfrak{a}_3)$ *be* $\mathfrak{a}$*-spaces.*

(a) *If* $f : X \to Y$ *and* $g : Y \to Z$ *are both* $\mathfrak{a}$*-continuous, then* $g \circ f : X \to Z$ *is* $\mathfrak{a}$*-continuous.*
(b) *If* $f : X \to Y$ *is* $\mathfrak{a}$*-semi-continuous and* $g : Y \to Z$ *is continuous, then* $g \circ f$ *is* $\mathfrak{a}$*-semi-continuous.*

Proof. (a) Let $W \in \tau_{\mathfrak{a}_3}$. Since g is $\mathfrak{a}$-continuous, $g^{-1}(W) \in \tau_{\mathfrak{a}_2}$. Since f is $\mathfrak{a}$-continuous, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W) \in \tau_{\mathfrak{a}_1}$.

(b) Let $W \in \tau_3$. Since g is continuous, $g^{-1}(W) \in \tau_2$. Since f is $\mathfrak{a}$-semi-continuous, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in a_1\mathrm{SO}(X)$. ☐

Theorem 11 (Decomposition of $\mathfrak{a}$-α-continuity)**.** *Let* $f : (X, \tau_1, \mathfrak{a}_1) \to (Y, \tau_2, \mathfrak{a}_2)$ *be a function.*

(a) *If* f *is* $\mathfrak{a}$*-*α*-continuous, then* f *is both* $\mathfrak{a}$*-semi-continuous and* $\mathfrak{a}$*-pre-continuous.*
(b) *The converse of (a) holds when* $\mathfrak{a}_1$ *is transitive.*

Proof. Part (a) follows from $a\alpha\mathrm{O}(X) \subseteq a\mathrm{SO}(X) \cap a\mathrm{PO}(X)$. For part (b), suppose $\mathfrak{a}_1$ is transitive (so $\mathrm{cl}_{\mathfrak{a}_1}$ is idempotent). Let $V \in \tau_2$ and $A = f^{-1}(V)$. From $\mathfrak{a}_1$-semi-continuity, $A \subseteq \mathrm{cl}_{\mathfrak{a}_1}(\mathrm{int}(A))$. From $\mathfrak{a}_1$-pre-continuity, $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}_1}(A))$. By idempotency, $\mathrm{cl}_{\mathfrak{a}_1}(A) \subseteq \mathrm{cl}_{\mathfrak{a}_1}(\mathrm{cl}_{\mathfrak{a}_1}(\mathrm{int}(A))) = \mathrm{cl}_{\mathfrak{a}_1}(\mathrm{int}(A))$. Taking interiors, $\mathrm{int}(\mathrm{cl}_{\mathfrak{a}_1}(A)) \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}_1}(\mathrm{int}(A)))$. Combined with $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}_1}(A))$, we get $A \subseteq \mathrm{int}(\mathrm{cl}_{\mathfrak{a}_1}(\mathrm{int}(A)))$, so $A \in a_1\alpha\mathrm{O}(X)$. ☐

Theorem 12 (Characterization of $\mathfrak{a}$-semi-continuity)**.** *Let* $f : (X, \tau_1, \mathfrak{a}_1) \to (Y, \tau_2)$ *be a function. The following are equivalent:*

(a) f *is* $\mathfrak{a}_1$*-semi-continuous.*
(b) *For every closed set* F *in* Y*,* $f^{-1}(F)$ *is* $\mathfrak{a}_1$*-semi-closed.*
(c) *For every* $x \in X$ *and every open set* V *in* Y *containing* $f(x)$*, there exists an* $\mathfrak{a}_1$*-semi-open set* U *containing* x *such that* $f(U) \subseteq V$*.*

Aura-Convergence of Sequences

Definition 15. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. A sequence* (x_n) *in* X *is said to* $\mathfrak{a}$-converge *to a point* $x \in X$*, written* $x_n \xrightarrow{\mathfrak{a}} x$*, if for every* $U \in \tau_{\mathfrak{a}}$ *with* $x \in U$*, there exists* $N \in \mathbb{N}$ *such that* $x_n \in U$ *for all* $n \geq N$*.*

Theorem 13. *Let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space and* (x_n) *a sequence in* X*.*

(a) *If* $x_n \to x$ *in* (X, τ)*, then* $x_n \xrightarrow{\mathfrak{a}} x$*.*
(b) *The converse of (a) need not hold.*
(c) *In an* $\mathfrak{a}$*-*T_2 *space (equivalently, an* $\mathfrak{a}$*-*T_1 *space; see Theorem 17),* $\mathfrak{a}$*-limits are unique.*

Proof. (a) Since $\tau_{\mathfrak{a}} \subseteq \tau$, every $\mathfrak{a}$-open neighborhood of x is also a τ-open neighborhood. If $x_n \to x$ in τ, then eventually x_n is in every τ-neighborhood of x, and hence in every $\tau_{\mathfrak{a}}$-neighborhood.

(b) Let $X = \{a, b\}$, $\tau = \mathcal{P}(X)$ (discrete), and $\mathfrak{a}(a) = \mathfrak{a}(b) = X$. Then, $\tau_{\mathfrak{a}} = \{\varnothing, X\}$. The constant sequence $x_n = a$ does not converge to b in τ, but $\tau_{\mathfrak{a}}$ has only X as a neighborhood of b, so $x_n \xrightarrow{\mathfrak{a}} b$. Hence, $\mathfrak{a}$-convergence need not imply τ-convergence, and limits need not be unique in non-T_0 spaces.

(c) Suppose $x_n \xrightarrow{\mathfrak{a}} x$ and $x_n \xrightarrow{\mathfrak{a}} y$ with $x \neq y$. Since the space is $\mathfrak{a}$-T_2, there exist disjoint $\mathfrak{a}$-open sets $U \ni x$ and $V \ni y$. By $\mathfrak{a}$-convergence, there exists N_1 such that $x_n \in U$ for $n \geq N_1$, and N_2 such that $x_n \in V$ for $n \geq N_2$. For $n \geq \max\{N_1, N_2\}$, $x_n \in U \cap V = \varnothing$, a contradiction. □

Example 10. *In* $(\mathbb{R}, \tau_u, \mathfrak{a}_\varepsilon)$ *with* $\mathfrak{a}_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$*, a set* U *is* $\mathfrak{a}$*-open if for every* $x \in U$*,* $(x - \varepsilon, x + \varepsilon) \subseteq U$*. The sequence* $x_n = 1/n$ *satisfies* $x_n \to 0$ *in* τ_u*; hence,* $x_n \xrightarrow{\mathfrak{a}} 0$ *by Theorem 13(a). The sequence* $y_n = (-1)^n$ *does not converge in* τ_u *and does not* $\mathfrak{a}$*-converge either, since for any* $\mathfrak{a}$*-open* $U \ni 1$*, the set* $(-1, 1 - \varepsilon) \cap U = \varnothing$ *blocks membership infinitely often.*

6. Aura-Separation Axioms

Definition 16. *An* $\mathfrak{a}$*-space* $(X, \tau, \mathfrak{a})$ *is called:*

(a) $\mathfrak{a}$-T_0 *if for every pair of distinct points* $x, y \in X$*, there exists an* $\mathfrak{a}$*-open set* $U \in \tau_{\mathfrak{a}}$ *such that either* $x \in U, y \notin U$ *or* $y \in U, x \notin U$*.*

(b) $\mathfrak{a}$-T_1 *if for every pair of distinct points* $x, y \in X$*, there exist* $\mathfrak{a}$*-open sets* $U, V \in \tau_{\mathfrak{a}}$ *such that* $x \in U, y \notin U$ *and* $y \in V, x \notin V$*.*

(c) $\mathfrak{a}$-T_2 *(or* $\mathfrak{a}$-Hausdorff*) if for every pair of distinct points* $x, y \in X$*, there exist disjoint* $\mathfrak{a}$*-open sets* $U, V \in \tau_{\mathfrak{a}}$ *such that* $x \in U$ *and* $y \in V$*.*

Theorem 14. *For any* $\mathfrak{a}$*-space,* $\mathfrak{a}$-$T_2 \Rightarrow \mathfrak{a}$-$T_1 \Rightarrow \mathfrak{a}$-$T_0$*. Moreover,* T_2 *for* (X, τ) *does not imply* $\mathfrak{a}$-T_2 *for* $(X, \tau, \mathfrak{a})$ *since* $\tau_{\mathfrak{a}} \subseteq \tau$ *and a coarser topology may fail to separate points.*

Example 11. *Let* $X = \{a, b, c\}$*,* $\tau = \mathcal{P}(X)$ *(discrete, hence* T_2*). Define* $\mathfrak{a}(a) = \mathfrak{a}(b) = \mathfrak{a}(c) = X$*. Then,* $\tau_{\mathfrak{a}} = \{\varnothing, X\}$*, which cannot separate any two points. Hence,* $(X, \tau, \mathfrak{a})$ *is not* $\mathfrak{a}$-T_0*.*

Theorem 15. *An* $\mathfrak{a}$*-space* $(X, \tau, \mathfrak{a})$ *is* $\mathfrak{a}$-T_1 *if and only if for every* $x \in X$*,* $\{x\}$ *is* $\mathfrak{a}$*-closed (i.e.,* $X \setminus \{x\} \in \tau_{\mathfrak{a}}$*).*

Proof. $(\Rightarrow)$: Let $y \neq x$. By $\mathfrak{a}$-T_1, there exists $U_y \in \tau_{\mathfrak{a}}$ with $y \in U_y$ and $x \notin U_y$. Then, $X \setminus \{x\} = \bigcup_{y \neq x} U_y \in \tau_{\mathfrak{a}}$.

$(\Leftarrow)$: Let $x \neq y$. Then $X \setminus \{y\} \in \tau_{\mathfrak{a}}$ contains x but not y, and $X \setminus \{x\} \in \tau_{\mathfrak{a}}$ contains y but not x. □

Theorem 16. *An* $\mathfrak{a}$*-space* $(X, \tau, \mathfrak{a})$ *is* $\mathfrak{a}$-T_0 *if and only if for every pair of distinct points* $x, y \in X$*, at least one of* $y \notin \mathfrak{a}(x)$ *or* $x \notin \mathfrak{a}(y)$ *holds.*

Proof. $(\Rightarrow)$: Let $(X, \tau, \mathfrak{a})$ be $\mathfrak{a}$-T_0 and let $U \in \tau_{\mathfrak{a}}$ separate x, y, say $x \in U$ and $y \notin U$. Since U is $\mathfrak{a}$-open and $x \in U$, $\mathfrak{a}(x) \subseteq U$. Since $y \notin U$, it follows that $y \notin \mathfrak{a}(x)$.

$(\Leftarrow)$: Suppose $y \notin \mathfrak{a}(x)$. Then, $\mathfrak{a}(x) \subseteq X \setminus \{y\}$, so x belongs to the set $U = \{w \in X : \mathfrak{a}(w) \subseteq X \setminus \{y\}\}$. By construction U is $\mathfrak{a}$-open. Moreover, $y \notin U$ because $y \in \mathfrak{a}(y)$ implies $\mathfrak{a}(y) \nsubseteq X \setminus \{y\}$. The case $x \notin \mathfrak{a}(y)$ is analogous. □

Example 12 (Granular separation on $\mathbb{R}$)**.** *Consider* $(\mathbb{R}, \tau_u, \mathfrak{a}_\varepsilon)$ *with* $\varepsilon > 0$*. For distinct* $x, y \in \mathbb{R}$ *with* $|x - y| > 2\varepsilon$*, the sets* $U = \left(-\infty, \frac{x+y}{2} - \varepsilon\right)$ *and* $V = \left(\frac{x+y}{2} + \varepsilon, +\infty\right)$ *are disjoint* $\mathfrak{a}$*-open* *sets separating x and y. Hence, $(\mathbb{R}, \tau_u, \mathfrak{a}_\varepsilon)$ separates points that are more than 2ε apart, but not those within distance 2ε—separation depends on the "granularity" determined by ε.*

Theorem 17. *In an $\mathfrak{a}$-space $(X, \tau, \mathfrak{a})$, the axioms $\mathfrak{a}$-T_1 and $\mathfrak{a}$-T_2 are equivalent. More precisely, the following are equivalent:*

(a) $(X, \tau, \mathfrak{a})$ *is* $\mathfrak{a}$-T_2.
(b) $(X, \tau, \mathfrak{a})$ *is* $\mathfrak{a}$-T_1.
(c) $\mathfrak{a}(x) = \{x\}$ *for every* $x \in X$.
(d) $\tau_{\mathfrak{a}} = \mathcal{P}(X)$ *(the discrete topology).*

This equivalence contrasts sharply with classical topology, where $T_1 \not\Rightarrow T_2$ in general.

Proof. $(a) \Rightarrow (b)$: Immediate from Definition 16.

$(b) \Rightarrow (c)$: By Theorem 15, $\mathfrak{a}$-T_1 implies $X \setminus \{y\} \in \tau_{\mathfrak{a}}$ for every $y \in X$. For any $z \neq y$, since $X \setminus \{y\}$ is $\mathfrak{a}$-open and $z \in X \setminus \{y\}$, we have $\mathfrak{a}(z) \subseteq X \setminus \{y\}$. In particular, $y \notin \mathfrak{a}(z)$ for all $z \neq y$. Since $z \in \mathfrak{a}(z)$ by (1), we conclude $\mathfrak{a}(z) = \{z\}$ for every $z \in X$.

$(c) \Rightarrow (d)$: If $\mathfrak{a}(x) = \{x\}$, then $A \in \tau_{\mathfrak{a}}$ iff $\{x\} \subseteq A$ for every $x \in A$, which is trivially true. Hence, $\tau_{\mathfrak{a}} = \mathcal{P}(X)$.

$(d) \Rightarrow (a)$: If $\tau_{\mathfrak{a}} = \mathcal{P}(X)$, then for any distinct x, y, the sets $\{x\}$ and $\{y\}$ are disjoint $\mathfrak{a}$-open sets separating x and y. ☐

Remark 4. *The equivalence $\mathfrak{a}$-$T_1 \Leftrightarrow \mathfrak{a}$-$T_2$ reflects the special nature of the aura framework. In classical topology, the Sierpiński space is T_0 but not T_1, and $\mathbb{R}/\mathbb{Q}$ is T_1 but not T_2. In the aura setting, however, any $\mathfrak{a}$-T_1 space forces every scope function to collapse to a singleton, immediately yielding the discrete $\mathfrak{a}$-topology and hence $\mathfrak{a}$-T_2.*

Definition 17. *An $\mathfrak{a}$-space $(X, \tau, \mathfrak{a})$ is called* $\mathfrak{a}$-regular *if for every $\mathfrak{a}$-closed set F and every point $x \notin F$, there exist disjoint $\mathfrak{a}$-open sets U, V such that $x \in U$ and $F \subseteq V$.*

Proposition 4. *If $\mathfrak{a}$ is discrete (i.e., $\mathfrak{a}(x) = \{x\}$ for all x), then $(X, \tau, \mathfrak{a})$ is $\mathfrak{a}$-T_2 and $\mathfrak{a}$-regular.*

7. Comparison with Existing Auxiliary Structures

In this section, we compare the aura framework systematically with the main auxiliary structures that have been used to enrich topological spaces.

7.1. Structural Comparison

Table 1 summarizes the key differences.

Table 1. Comparison of auxiliary structures in topology.

Feature	Ideal	Grill	Primal	Aura
Auxiliary data	$\mathcal{I} \subseteq \mathcal{P}(X)$	$\mathcal{G} \subseteq \mathcal{P}(X)$	$\mathcal{P} \subseteq \mathcal{P}(X)$	$\mathfrak{a} : X \to \tau$
Point-dependent	No	No	No	Yes
Closure type	Kuratowski	Kuratowski	Kuratowski	Čech
Idempotent	Yes	Yes	Yes	No (in general)
Produces topology	τ^* (finer)	τ_G (finer)	τ_P (finer)	$\tau_{\mathfrak{a}}$ (coarser)
Direction vs. τ	$\tau \subseteq \tau^*$	$\tau \subseteq \tau_G$	$\tau \subseteq \tau_P$	$\tau_{\mathfrak{a}} \subseteq \tau$
Topology chain	Single	Single	Single	$\tau_{\mathfrak{a}}^\infty = \tau_{\mathfrak{a}} \subseteq \tau$
$T_1 \Leftrightarrow T_2$	No	No	No	**Yes**

The most fundamental distinction is directional: ideals, grills, and primals all produce topologies that are finer than τ, while the aura topology $\tau_{\mathfrak{a}}$ is coarser. This has profound consequences: in the aura setting, there are fewer open sets available for covers (making

$\mathfrak{a}$-compactness easier to achieve) and fewer ways to separate points (making $\mathfrak{a}$-separation harder), which is the exact opposite of what happens with ideals.

7.2. Operator-Level Comparison

Theorem 18. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space with local function* $A^* = \{x \in X : U \cap A \notin \mathcal{I} \text{ for all } U \in \tau(x)\}$, *and let* $(X, \tau, \mathfrak{a})$ *be an* $\mathfrak{a}$*-space. Then,*

(a) *The ideal closure* $\mathrm{cl}^*(A) = A \cup A^*$ *satisfies* $\mathrm{cl}^*(A) \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$ *in general.*
(b) *The reverse inclusion* $\mathrm{cl}_{\mathfrak{a}}(A) \subseteq \mathrm{cl}^*(A)$ *does not hold universally.*

Proof. (a) The inclusion $\mathrm{cl}^*(A) \subseteq \mathrm{cl}(A)$ is well-known: $A^* \subseteq \mathrm{cl}(A)$ since $x \in A^*$ implies every open neighborhood of x meets A. The inclusion $\mathrm{cl}(A) \subseteq \mathrm{cl}_{\mathfrak{a}}(A)$ is Theorem 1(e).

(b) Take $X = \{a, b, c\}$, $\tau = \mathcal{P}(X)$, $\mathcal{I} = \mathcal{P}(X)$ (improper ideal), $\mathfrak{a}(a) = \{a, b\}$, $\mathfrak{a}(b) = \{b\}$, $\mathfrak{a}(c) = \{c\}$, and $A = \{b\}$. Then, $A^* = \varnothing$, so $\mathrm{cl}^*(A) = \{b\}$. But $\mathrm{cl}_{\mathfrak{a}}(A) = \{a, b\}$ since $\mathfrak{a}(a) \cap \{b\} \neq \varnothing$. Hence, $\mathrm{cl}_{\mathfrak{a}}(A) \not\subseteq \mathrm{cl}^*(A)$. □

8. Conclusions

We have introduced a new auxiliary structure for topological spaces: the scope function $\mathfrak{a} : X \to \tau$ with $x \in \mathfrak{a}(x)$, and the resulting aura topological space $(X, \tau, \mathfrak{a})$. This simple axiom—a point-to-open-set assignment—proved to be surprisingly productive.

On the operator side, the aura-closure $\mathrm{cl}_{\mathfrak{a}}$ is an additive Čech closure operator whose failure of idempotency leads naturally to the transfinite iteration $\mathrm{cl}_{\mathfrak{a}}^{\infty}$. We proved that $\mathrm{cl}_{\mathfrak{a}}^{\infty}$ generates exactly the same topology as $\mathrm{cl}_{\mathfrak{a}}$, yielding the clean equalities $\tau_{\mathfrak{a}}^{\infty} = \tau_{\mathfrak{a}} \subseteq \tau$. This rigidity—the collapse of the topology chain to $\tau_{\mathfrak{a}}$—has no analogue in the ideal, grill, or primal frameworks.

On the open-set side, combining $\mathrm{cl}_{\mathfrak{a}}$ with the standard interior produces five generalized open-set classes whose hierarchy we determined completely and separated non-coinciding classes by explicit examples. The corresponding continuity notions admit decomposition theorems extending the classical results of Levine and Mashhour. The notion of $\mathfrak{a}$-convergence of sequences was introduced, and its relation to classical convergence was established.

On the separation side, we proved that the axioms $\mathfrak{a}$-T_1 and $\mathfrak{a}$-T_2 are equivalent in any $\mathfrak{a}$-space—a sharp contrast with classical topology where these axioms are genuinely distinct.

Several directions remain for future work. In a companion paper [34], we investigate compactness and connectedness in aura spaces, including a Tychonoff-type theorem for transitive aura spaces and product/subspace constructions. Combining the aura function with an ideal opens the way to a hybrid local function and a topology interpolating between $\tau_{\mathfrak{a}}$ and τ^*. On the applied side, connections with rough set theory, digital topology, and information systems appear promising.

Funding: This research received no external funding.

Data Availability Statement: No data were used for the research described in the article. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflicts of interest.

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