



Mathematical Working Spaces in the mathematical modeling cycle: the case of preservice elementary mathematics teachers

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Abstract

Mathematical modeling is a dynamic process through which real-life situations are explored and explained via mathematical knowledge and reasoning. The theory of Mathematical Working Spaces (MWS) offers a robust framework for analyzing the cognitive process involved in modeling, emphasizing the roles of components of representation, artifacts, and theoretical reference. This study investigates the mathematical works of preservice mathematics teachers as they engage in a modeling task, using the MWS as an analytical lens. Adopting a qualitative case study design, data were collected from 34 preservice teachers selected via criterion sampling. Sources included written responses to guided exploratory questions and video recordings of group work. The modeling actions were examined through descriptive analysis grounded in the components of MWS. The findings reveal that the diversity and depth of actions initiated in the early phases of the modeling cycle—particularly within the genesis processes—are critical for the successful completion of the task. The study highlights that engaging with the genesis processes meaningfully, beyond merely operating at the epistemological level, supports richer mathematical reasoning and more complete modeling outcomes.

Keywords Mathematical Working Spaces · Mathematical work · Mathematical modeling · Preservice elementary mathematics teachers

1 Introduction

Mathematical modeling has a significant position in mathematics curricula as a competency that requires improvement at all levels. Modeling bridges mathematics and real-world contexts and involves cognitive and epistemological processes that enable students to construct, verify, and communicate mathematical knowledge. Therefore, modeling is considered a central component of mathematical thinking and learning (Blum & Leiß, 2007).

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However, there are significant challenges for future mathematics teachers who will transfer modeling to the classroom. While preservice mathematics teachers are expected to design and implement learning activities that develop modeling competencies, research has shown that many preservice teachers struggle to select appropriate representations, coordinate tools, and construct valid justifications in the classroom (Stillman et al., 2013). Considering that preservice mathematics teachers will teach modeling in their future classrooms and that literature reports persistent difficulties in modeling pedagogy for teacher candidates, it is essential to investigate the process of solving modeling tasks.

The theory of Mathematical Working Spaces (MWS) provides a comprehensive framework for analyzing mathematical work by considering epistemological and cognitive dimensions (Kuzniak, 2022). MWS explains how representations, artifacts, and theoretical references interact through semiotic, discursive, and instrumental genesis processes to build mathematical understanding. In this regard, MWS serves as a theoretical framework for analyzing the cognitive steps of modeling, revealing how these steps are shaped by genesis processes and epistemological components. Recent studies have highlighted that integrating MWS with modeling can deepen analysis of mathematical modeling processes (Derouet et al., 2017; Verdugo Hernández et al., 2023). Lagrange et al. (2022) also emphasized the potential of MWS to analyze modeling activities, although empirical studies in this integration remain scarce. Therefore, using MWS to examine modeling allows us to observe how cognitive processes such as visualization, construction, and justification are guided by tools, representations, and theoretical knowledge, making invisible aspects of modeling visible and analyzable.

This study aims to investigate how preservice mathematics teachers utilize the components of MWS theory when solving tasks involving mathematical modeling. By examining how modeling competencies are enacted through the lens of MWS, this research contributes to clarifying conceptual ambiguities in the integration of modeling and MWS in teacher education. Considering that preservice teachers play a critical role in realizing curriculum objectives related to modeling, it is necessary to understand how they conceptualize and construct modeling processes within this theoretical framework. Specifically, this study addresses the following research questions:

- 1) What are the modeling competencies of preservice mathematics teachers in the mathematical modeling process?
- 2) What MWS components are used in the mathematical works of preservice mathematics teachers while solving the mathematical modeling task, and how are the interactions between the components?

These two questions are interrelated, as the modeling competencies observed are analyzed through the lens of MWS components to reveal the cognitive and epistemological processes underlying their solutions.

2 Theoretical framework

2.1 Mathematical Working Spaces (MWS)

MWS is a framework that interprets the mathematical work performed by students or teachers while solving mathematical tasks in instructional settings (Kuzniak & Nechache,

2021; Kuzniak, et al., 2016a, 2016b; Kuzniak et al., 2022). Its overarching objective is to elucidate how the interactions among different components within mathematical work give rise to learning and understanding (Kuzniak, 2022; Nechache & Gómez-Chacon, 2022). MWS considers both epistemological dimensions—such as representations, artifacts, and theoretical references—and cognitive dimensions, including visualization, construction, and proof (Kuzniak & Richard, 2014). Therefore, MWS is regarded as a significant instrument for analyzing mathematical tasks and comprehending the processes involved (Lagrange et al., 2022).

The MWS framework is represented as a triangular prism. The epistemological plane (lower base) consists of representations, artifacts, and theoretical referential components, which are shaped by mathematical paradigms and guide the conduct of mathematical work. The cognitive plane (upper base) consists of visualization, construction, and proof components, adapted from Duval (2005), emphasizing the identification of cognitive processes through observing mathematical actions. The interactions between these components are described by genesis processes: semiotic genesis (linking representations to visualization), instrumental genesis (connecting artifacts to construction), and discursive genesis (organizing reasoning and justification through theoretical references). Within the MWS framework, these interactions are represented by following combinations, referred to as vertical planes: semiotic-instrumental [Sem-Ins], semiotic-discursive [Sem-Dis], and instrumental-discursive [Ins-Dis] (Kuzniak, 2022) (Fig. 1).

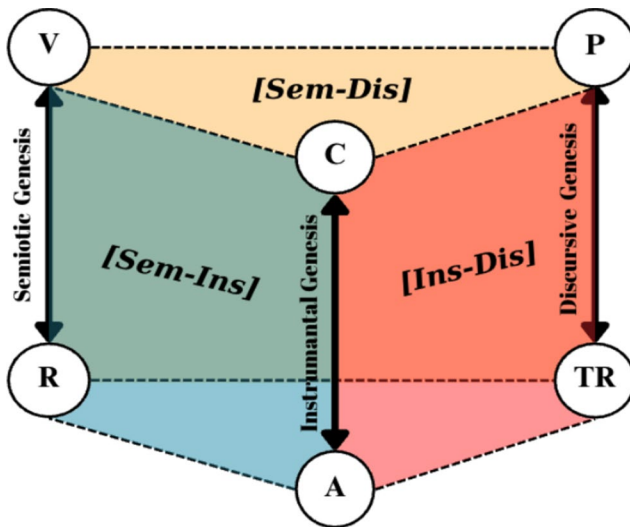


Fig. 1 MWS diagram and vertical planes (adapted from Kuzniak et al., (2016a, 2016b))

This framework enables the analysis of how students mobilize representations, tools, and theoretical knowledge to perform mathematical work, providing insights into the cognitive processes that shape mathematical understanding.

2.2 Mathematical modeling

Mathematical modeling is a central process in mathematics education that connects mathematics to real-world phenomena. It enables students to translate real situations into mathematical representations, solve problems, and interpret results back in the context (Blum & Leiß, 2007; Ledezma et al., 2023), as shown in Fig. 2. Modeling activities require selecting appropriate representations, mathematizing real situations, interpreting problem information, and justifying solutions (Lesh & Doerr, 2003). While it is not feasible to engage in modeling without mathematical representations, semiotic systems, tools, and reasoning processes must be coordinated (Lagrange et al., 2022).

While presented as a sequential and circular process, in practice, modeling work rarely follows this structure linearly. Instead, it often involves iterative, recursive, and non-linear pathways, as students revisit earlier stages based on new insights, errors, or reflections (Blum & Leiß, 2007); Reyes Avendaño, 2020). Recognizing this iterative nature is critical when analyzing modeling processes, as it reveals how learners adapt and restructure their cognitive strategies in response to emerging challenges and partial results.

Among various modeling cycles, the model proposed by Blum and Leiß (2007) is widely used because it focuses on the cognitive actions undertaken during modeling. This cycle is idealized structure that guides the analysis of real-world problems in educational contexts. Analyzing the modeling cycle through the lens of MWS reveals how cognitive and epistemological components interact during modeling tasks, providing a comprehensive understanding of how preservice teachers engage in modeling as a learning and teaching process.

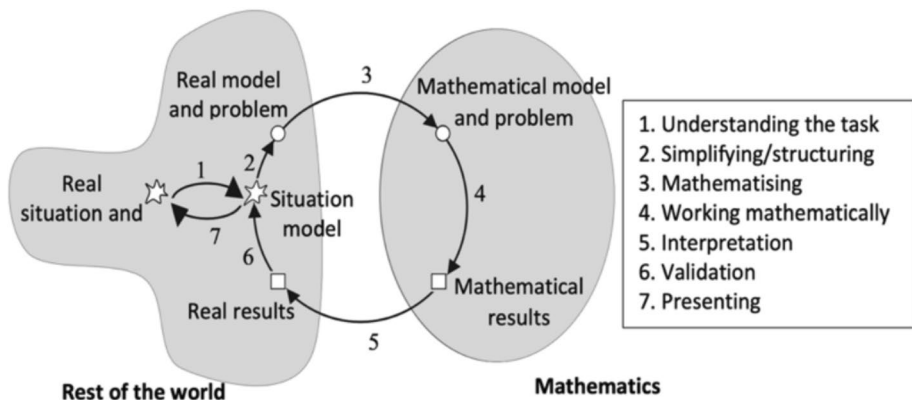


Fig. 2 Mathematical modeling cycle (Blum & Leiß, 2007)—reproduced by Kotze (2017, p. 193)

3 Method

3.1 Research model

In this study, the case study method was taken as a basis, which is a qualitative research design that involves the detailed examination of one or more situations through data obtained from various information sources (Yıldırım & Şimşek, 2021). A single nested case design was adopted, as it focuses on the actions taken to solve a mathematical task. The objective of adopting this approach is to ascertain the actions undertaken during mathematical work, to elucidate the actions with explanatory statements, and to evaluate the mathematical work by assessing the explanatory statements.

3.2 Research group

Preservice teachers were chosen as they will implement modeling in their future classrooms and are key agents for integrating modeling competencies into practice. The study group was selected via criterion sampling, which is a purposive sampling method that involves in-depth analysis through various cases (Büyüköztürk et al., 2008). Individuals who met a predetermined criterion constituted the study group. The participants were selected on the basis of the criterion that they had completed a mathematical modeling course and had experience using technology in mathematics education.

During their undergraduate education, the preservice teachers in the study group were enrolled in material design, algorithm, geometry, and algebra teaching courses. Based on the course content, they had experience with web 2.0 applications, dynamic geometry software, block-based programming environments, and both virtual and physical manipulatives. These characteristics ensured that participants could meaningfully engage with both the components of the modeling cycle and the relevant technological tools during the task involving mathematical modeling activity.

The study group consisted of 34 preservice elementary mathematics teachers studying at a state university in western Türkiye who had successfully completed courses on mathematical modeling and the integration of technology in mathematics education. The preservice teachers were divided into groups of three or four members in order to complete the mathematical task. In forming the groups, heterogeneous groups were formed on the basis of their performance in the mathematical modeling course, as detailed in Table 1. The scores presented are grades based on a 0 to 4 scale, where 0 indicates failure and 4 indicates excellent performance. These scores reflect students' achievement in the mathematical modeling course.

Table 1 Characteristics of the groups

Groups	G1	G2	G3	G4	G5	G6	G7	G8	G9
The average score on the mathematical modeling course	3.33	3	2.75	3.25	2.62	3.12	2.87	3.37	3.5

3.3 Data collection tools and process

To design the study, national curriculum documents (Ministry of National Education [MoNE], 2018, 2024) and the undergraduate field education courses in elementary mathematics teaching (Balıkesir University, 2022) were reviewed. In this context, developing a mathematical task within the notion of area measurement was determined. This concept provides rich opportunities for exploring different geometric shapes and solution strategies across various educational levels. Mathematical modeling, aimed at understanding real-life situations and developing solutions, incorporates tangible objects (Blum & Leiß, 2007; Ekol & Greenop, 2023; Jablonski, 2023). The tasks developed through this process facilitate the demonstration of both calculation skills and higher-order cognitive abilities, such as problem-solving and planning (Herget & Torres-Skoumal, 2007). Therefore, the design of the task incorporated core elements of model-eliciting activities (Bukova Güzel, 2016; Chamberlin & Moon, 2005; Tekin Dede & Bukova Güzel, 2018). The model-eliciting activity is delineated in Table 2, while the process of developing the model-eliciting activity is outlined in Table 3.

Table 3 presents the model-eliciting activity development process, showing each component or principle was operationalized in the task design. The introductory text and readiness questions support understanding and simplifying, the reality and model construction principles guide mathematization, self-assessment and construct certification relate to working mathematically and validation, while generalization, effective prototype, and solution presentation correspond to interpreting, communicating, and reapplying results. The final column details the concrete steps taken to incorporate these principles, ensuring consistency with both model-eliciting activity standards and the modeling cycle.

Table 2 Developed model-eliciting activity

According to the agreement signed between Galatasaray Sports Club and Enerjisa Energy, the solar panels installed on the roof of the stadium will generate 4637 MWh of electricity per year, equivalent to the consumption of approximately 2000 households. With more than 10 thousand solar panels installed on the roof of the stadium, an installed power capacity of 4.1 MW has been reached, making it the largest capacity solar power plant among the stadiums in the world. The facility, which will generate electricity from 100% green energy sources, will contribute to nature by preventing the emission of 3250 tons of carbon dioxide every year. Determine the area covered by the solar panels installed on the stadium roof with this agreement that will contribute to the environment, society, and the financial sustainability of Turkish sports.



The aerial photograph was obtained from Google Earth

Table 3 Model-eliciting activity development process

Components/principles		Properties of components and principles	Process of creating the task
Introductory text		It consists of a reading passage written to generate interest and give an idea of the context of the task.	An introductory text was created based on the information obtained from the internet (Galatasaray Sports Club, n.d.; Enerjisa Energy, n.d.).
Readiness questions		It includes simple comprehension questions about the introductory text.	To understand the information in the introductory text and the task, questions were given with worksheet.
Problem situation	Reality P.	It should involve situations that are meaningful in real life.	A problem was created to determine the area covered by the solar panels on the roof of the stadium since it includes that may be meaningful in the context of real life.
	Model construction P.	The task should require creating a model.	In the mathematical work, it was thought that a mathematical model could be created by considering the geometric shapes.
	Self-assessment P.	It should be possible to decide whether the solutions are valid through group discussions, and there should be no need to get help.	The mathematical work was planned to be carried out with group work. In this direction, it was aimed to exchange ideas, to carry out the group work without help, to interpret whether the mathematical result is appropriate in the context of real life.
	Construct certification P.	The solution to the task should be expressed in detail.	The actions performed by the preservice teachers were asked to be expressed aloud and recorded with a camera in order to avoid missing data. In addition, the preservice teachers were asked to express the problem solution in writing in detail with the group report.
	Construct generalization P.	The mathematical model should be generalizable and usable in similar situations.	It was taken into consideration that the mathematical models developed for the solution of the task could be generalized to similar situations and serve as an example that could be used in similar situations.
	Effective prototype P.	The mathematical model should remain valid in similar situations.	
Presentation of solutions		Solutions should be discussed in class.	After the group work, a discussion environment in the classroom was planned. Each group presented their mathematical work in the classroom.

In this task, preservice teachers were expected to begin by identifying an appropriate reference object within the aerial photograph to establish a real-world scale. Following this, they were expected to calculate the scale ratio between the dimensions in the photograph and their actual measurements. Using this ratio, they were expected to model the stadium roof with geometric shapes such as rectangles, triangles, and circles to approximate the area covered by the solar panels. Subsequently, they were expected to perform the necessary area calculations based on the geometric models they constructed. Finally, preservice teachers were expected to convert their calculated results into appropriate units

and interpret these values in the real-world context, articulating their conclusions regarding the environmental and societal contributions of the solar panel installation.

Furthermore, when analyzed through the lens of MWS theory, this task was designed to mobilize specific genesis processes during mathematical work. In particular, identifying a reference object and constructing it within GeoGebra was expected to activate instrumental genesis by engaging technological tools to create accurate measurements. Calculating scale ratios and modeling with geometric shapes were anticipated to involve semiotic genesis, as preservice teachers would need to generate and transform representations linking real-world measures to mathematical abstractions. Finally, interpreting the results and justifying area calculations in context were designed to elicit discursive genesis, requiring theoretical reasoning and validation of solutions. This analysis of the expected solution process in relation to MWS components provides a conceptual basis for comparing the anticipated and actual genesis processes enacted by preservice teachers during modeling.

To accompany the model-eliciting activity, guided exploratory questions were designed to reflect the phases of the mathematical modeling cycle and the components of MWS. These questions were refined through expert feedback and piloted with a separate group. The preservice teachers were divided into groups of three or four members in order to complete the mathematical task. Participants were instructed to use technological tools, verbalize their thought processes in detail, and record their group work in written reports. Video and audio recordings were also made to capture the full process. This multimodal data collection aimed to ensure consistency and triangulate observation data with worksheet responses. The modeling task and the group work lasted approximately one hour per group.

3.4 Data analysis

Given the integration of a modeling activity within the task, the Modeling Competencies Assessment Rubric (MCAR) developed by Tekin Dede (2015) was employed to evaluate the mathematical works. The MCAR assesses competencies by addressing each stage of the modeling cycle and determining the level of proficiency demonstrated in each stage. According to the rubric, scores of 0–6 indicate no competence, scores of 7–12 reflect partial competence, scores of 13–21 represent acceptable competence, and scores of 22–25 signify high competence.

Considering the aim of the study, which was to examine in detail the MWS components and genesis processes within the mathematical work carried out during modeling tasks, a focused and in-depth qualitative analysis was prioritized over analyzing all groups. Therefore, based on expert opinions, it was decided to analyze only the four groups with the highest modeling proficiency, as determined by their MCAR scores. Groups scoring below 7 were excluded because such scores indicate insufficient modeling competence. This selection strategy enabled the study to explore diverse modeling strategies among groups demonstrating high proficiency and provide generalizable insights by comparing groups with similar levels of modeling competence.

To analyze preservice teachers' mathematical work in line with MWS theory, the analysis method defined by Kuzniak and Nechache (2021) was employed. This method combines a top-down classification of modeling actions with a bottom-up interpretation through MWS components, enabling a comprehensive understanding of cognitive and epistemological processes.

The analysis was conducted in three phases. First, a top-down analysis was carried out, classifying the actions performed in solving the mathematical task based on the modeling cycle. The understanding and simplification stages were considered together due to their

Table 4 MWS components and descriptors

Components		Descriptor
Semiotic Genesis	Representations	Using mathematical objects, symbols, and shapes and associating them with each other.
	Visualization	Expressing mathematical representations in different ways in the actions performed, expressing thoughts verbally or in writing.
Instrumental Genesis	Artifact	Use of tools such as geometric drawing tools or software.
	Construction	Actions initiated by artifacts; constructions realized with dynamic geometry software.
Semiotic Genesis	Theoretical References	Using the properties of a mathematical concept.
	Proof	Reasoning process, justification.
Vertical Planes	[Sem-Ins]	Using artifacts in the process of visualizing and representing mathematical objects, Making the artifact available so that the mathematical representation can be used, Determination and visualization of shapes using dynamic geometry software.
	[Ins-Dis]	Verification and justification of the reasoning process with artifacts, Enriching theoretical references with artifacts, Realization of the construction process with the arguments in the theoretical reference.
	[Sem-Dis]	Making logical explanations with reasoning, Visualization using theoretical references, transforming them into different forms of representation, Using the visualization process in coordination with reasoning.

similar actions, as were the interpretation and validation stages because they can occur simultaneously. Transcribed mathematical works were examined, and each section was assigned to a stage of the modeling cycle. Second, a bottom-up analysis was conducted by interpreting the actions in line with MWS components and descriptors. Finally, the findings from both phases were integrated to determine whether the mathematical work was complete and to identify the genesis or vertical plane limitations. To support this analysis, the circulation of actions within the MWS diagram was reported using a visual framework. Table 4 summarizes the MWS components and descriptors used to interpret the modeling actions in this study (Henríquez Rivas et al., 2021; Espinoza-Vásquez et al., 2024).

The coding process was carried out independently by two coders, and intercoder reliability, calculated using percentage agreement, exceeded 80%, which is considered sufficient for consistency. Any discrepancies were discussed and resolved by consensus, and when necessary, a third field expert was consulted to reach the final decision. The validity of the analysis was enhanced through data triangulation, expert consultation, and the inclusion of direct quotations and visual artifacts in the presentation of findings (Erlandson et al., 1993; Miles & Huberman, 1994; Yıldırım & Şimşek, 2021).

4 Results

Table 5 summarizes the MCAR scores, which were used to assess preservice teachers' modeling competencies before conducting in-depth analysis of their mathematical work.

Table 5 MCAR scores

Groups	Mathematical modeling cycle						Total score
	Understanding the task	Simplifying	Mathematizing	Working mathematically	Interpretation	Validation	
G1	3	2	2	2	1	1	11
G2	3	3	3	3	4	0	16
G5	3	3	3	2	0	0	11
G7	3	3	4	3	2	2	17

A comparison of the groups reveals that G7 achieved the highest overall score (17 points), demonstrating particular strength in the mathematizing and validation stages. G2 also performed at a proficient level (16 points), notably achieving the highest validation score among all groups. In contrast, G1 and G5 received lower scores (11 points), reflecting limited competence particularly in the interpretation and validation stages, where both groups scored either 0 or 1 point. This suggests that while these groups were able to engage in early modeling stages, their solutions lacked adequate justification and communication.

According to the rubric criteria, scores between 7 and 12 indicate limited modeling competence, whereas scores from 13 to 21 signify an acceptable proficiency level. Therefore, G1 and G5 demonstrated partial modeling competence, while G2 and G7 exhibited acceptable levels of proficiency, indicating a stronger command of the modeling process.

These MCAR findings are important as they provide a preliminary evaluation of each group's modeling competencies before deeper analysis. They highlight specific strengths and weaknesses in preservice teachers' modeling processes, particularly emphasizing the need to support validation and presentation stages, which are essential for constructing robust mathematical models and communicating them effectively. Thus, the MCAR results provided a foundation for analyzing their mathematical work through the lens of MWS, enabling a comprehensive understanding of how competencies manifest in cognitive and epistemological processes.

4.1 Mathematical work of G1

G1 began by searching the internet to find a bird's-eye photograph of the stadium and the dimensions of the football field, which they then transferred into GeoGebra. They stated that the task could be solved by identifying geometric shapes within the image and selecting a reference object to establish scale. During this phase, they used GeoGebra tools such as point creation and length measurement, applied the mathematical concepts of ratio and perpendicularity, and utilized rounding and estimation skills. These actions demonstrate the activation of epistemological plane components, as they combined technological tools with conceptual understanding to prepare for mathematization.

The following discussion occurred among G1 while mathematizing the task:

PT2: I think it would be easier to calculate the panels separately. Let us consider the panels on top as rectangles.

PT1: How do we find the other panels?

PT2: We divide the panels and consider other sides as triangles.

This discussion revealed that they transferred the real-world task into the mathematical domain by decomposing the structure into simpler geometric components. Here, they activated discursive genesis (reasoning and justification based on theoretical references) as they planned to calculate areas using known geometric formulas.

During the working mathematically phase, G1 created a rectangular shape by constructing perpendicular intersecting line segments on solar panels in GeoGebra and determined its side lengths with the measurement tool. They proposed that triangle-like sections could be obtained by cutting the rectangular area appropriately and that these triangular areas should also be calculated. These actions illustrate instrumental genesis, as the preservice teachers used GeoGebra's construction and measurement tools to model and manipulate geometric shapes in the task. At the same time, semiotic genesis was evident through their creation and transformation of representations, as they visualized the panels, divided them into known geometric forms such as rectangles and triangles, and structured their approach based on these representations to facilitate calculation.

However, they could not fully complete the solution with this strategy. An alternative approach emerged in their subsequent discussion:

PT2: Look, the small rectangle has four by four panels, and the other has five by six panels. By using these rectangles, we can obtain the total area more easily.

PT3: Actually, we can find the area occupied by a panel and then proceed to the whole thing.

In this alternative strategy at second mathematization stage, they reasoned that by dividing the panels into smaller units, calculating the area of one panel, and scaling up, they could determine the total area more effectively. This process involved semiotic genesis, as the preservice teachers modeled the panels as rectangles to facilitate their calculations, and discursive genesis, as they applied reasoning based on the concepts of ratio and area to justify their approach. When they made calculations at second working mathematically stage, instrumental genesis occurred, as they used GeoGebra tools to measure lengths in the photograph and calculate areas based on real-world scaling, and this process involved discursive genesis, as they applied reasoning based on the theoretical references of ratio to justify their actions.

The group concluded their work by reflecting on their results, stating: "*If we persist in our operations with the initial method and contrast it with the result we obtained with this method, the precise result will be a value between these two results.*" This statement demonstrates discursive genesis at interpretation and validation phases as they attempted to validate their findings by comparing multiple methods. Figure 3 shows G1's interactions between modeling phases and MWS.

Overall, the analysis of G1's mathematical work, as illustrated in Fig. 3, shows that components of the epistemological plane were predominantly activated during the understanding and simplification stages, supporting their identification of geometric shapes and reference objects. Discursive genesis emerged clearly during mathematization, as they reasoned about decomposing the stadium into simpler shapes. The initial mathematically working phase corresponded to the [Sem-Ins] vertical plane. A repeated mathematization phase was situated in the [Sem-Dis] plane, followed by a repeated mathematically working phase associated with the [Ins-Dis] plane. These observations reveal that their limited activation of interpretation and validation phases suggests a need for further support in these areas to strengthen their overall modeling proficiency.

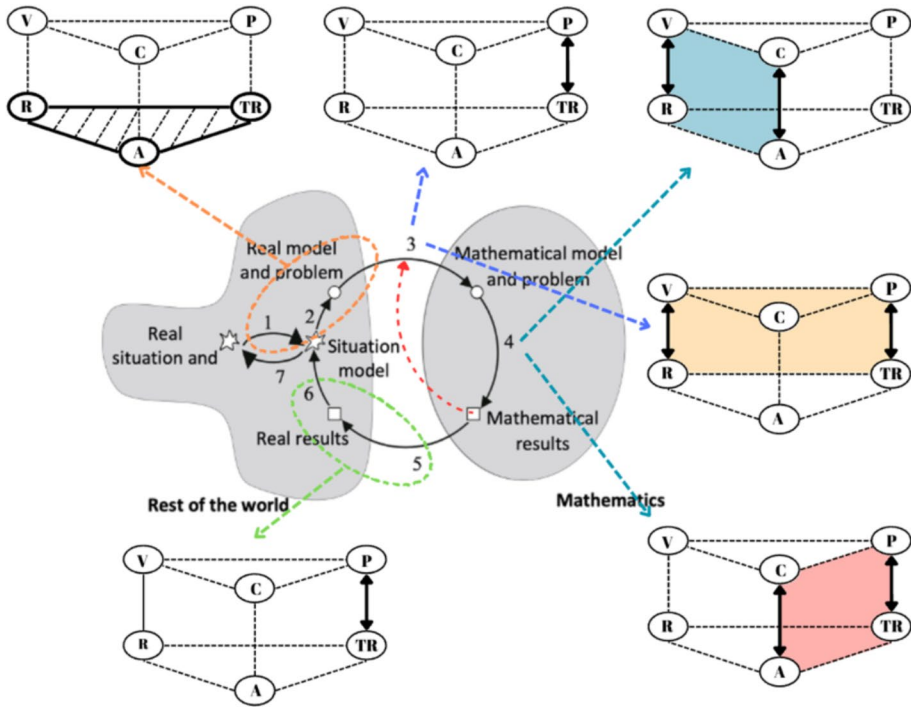


Fig. 3 Mathematical work of G1

4.2 Mathematical work of G2

G2 began by locating bird’s-eye-view photograph of the stadium on the internet and transferred it into GeoGebra. They identified a car in the photograph as a reference object, marking its ends with the point tool and connecting these points with the line segment tool to represent it geometrically. They then measured the length of this segment using the algebra window. At this stage, their use of GeoGebra tools (instrumental genesis: employing tools to construct representations) and selection of geometric representations (semiotic genesis: creating and interpreting visual representations) demonstrated effective integration of technology with conceptual understanding. They also proposed using geometric shapes such as rectangles and circles to solve the task.

The following discussion occurred among G2:

PT4: If we think of the whole stadium as a rectangle, we will calculate the area with a minimal error. If we consider as a circle, it seems that we will have an overestimated area.

PT5: It makes more sense to think rectangular.

PT4: Okay, let us consider the stadium’s shape as a rectangle. Then, we subtracted the area of the football field. We need lengths of the rectangle.

This discussion for the mathematization of the problem demonstrates discursive genesis as they evaluated different geometric approximations to model the stadium realistically.

Their decision to model the stadium as a rectangle shows semiotic genesis by selecting a representation that facilitated manageable calculations.

In the working mathematically phase, G2 constructed the sides of the rectangular model using the line segment tool and measured them via the algebra window, then expressed these dimensions through mathematical representations. Here, instrumental genesis was evident as they used GeoGebra tools to create and measure the rectangle, while semiotic genesis emerged as they visualized and structured the stadium model. They applied the scale ratio derived from the reference object to convert measurements into real-life dimensions, and their calculations were supported by explicit reasoning, reflecting discursive genesis through the application of geometric properties and ratio concepts.

Then, G2 reviewed their calculations and justified final decisions. Figure 4 presents GeoGebra-based calculations.

Besides, G2's reflective interpretation of their numeric results and the justification of their final decision corresponded to discursive genesis. The semiotic genesis was evident as G2 discussed the geometric shapes of the panes and used mathematical representations to support visualization. This reflection illustrates discursive genesis as they validated their results through comparative reasoning. Figure 5 shows G2's interactions between modeling phases and MWS.

Overall, G2 effectively utilized genesis processes throughout the modeling process. Their work showed strong engagement with the modeling cycle stages, particularly in mathematization and working mathematically phases, with only minor limitations observed in the interpretation and validation stages. In the case of G2, mathematical work predominantly took place in the [Sem-Ins] vertical plane during understanding the task and simplification stages, shifted to [Sem-Dis] vertical plane during the mathematization, and engaged all vertical planes of the MWS during the working mathematically phase. In the interpretation and validation phases, [Sem-Dis] was revisited, where discursive genesis became prominent as they justified and validated their results. These findings highlight

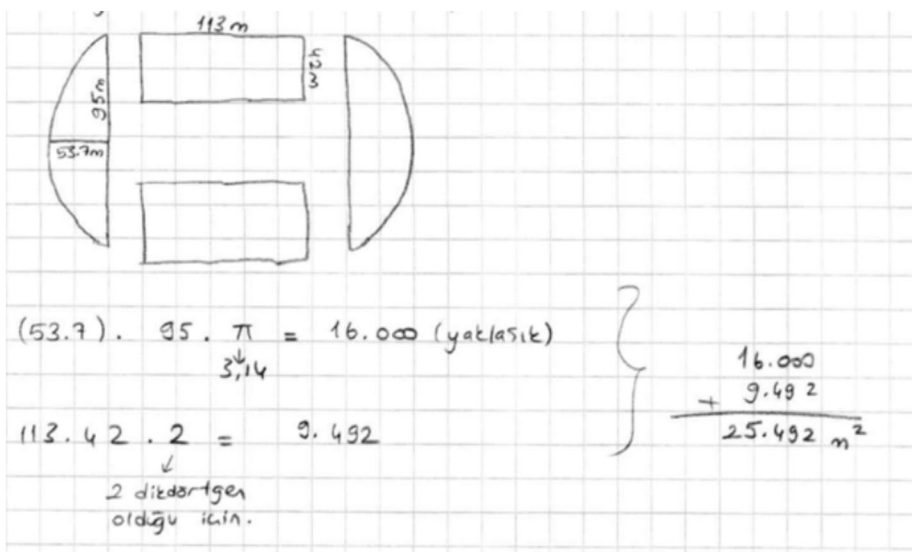


Fig. 4 Worksheet excerpt of G2

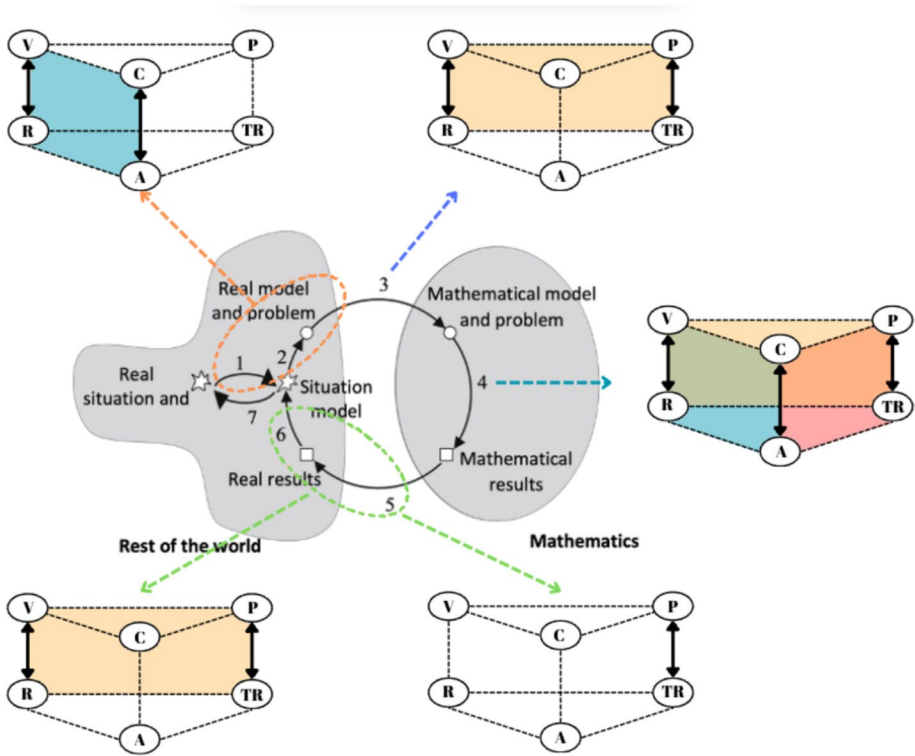


Fig. 5 Mathematical work of G2

that while G2 exhibited a high level of modeling proficiency overall, minor limitations remained in their validation and generalization processes, revealing which components were activated and which transitions were constrained within their modeling work.

4.3 Mathematical work of G5

In understanding the task and simplification stages, G5 proposed that an approximate result could be obtained through the use of GeoGebra. They obtained a bird’s-eye view photograph of the stadium via internet research and imported it into GeoGebra. They identified a reference object within the image, creating a line segment to represent it using the line segment tool and measuring its length via the algebra window. These actions demonstrate the activation of instrumental genesis and semiotic genesis as they modeled the reference object. They also considered geometric shapes such as ellipses and circles as possible representations and justified their selections based on shape properties of theoretical references.

Then, at the mathematization stage, G5 suggested modeling the stadium as a circle and determining the difference between the area of this circle and the area of the rectangular football field. To operationalize this, they constructed a line segment across the stadium image, used it as the diameter to create a circle, and delineated the football field boundaries

within it. Accordingly, a line segment was constructed across the stadium image, and its length was ascertained with the algebra window (Fig. 6). These actions illustrated instrumental genesis through the use of GeoGebra tools to construct geometric models and semiotic genesis by representing the stadium and field with geometric shapes to facilitate calculation.

During the working mathematically phase, G5 measured the reference object's length using the algebra window and discussed appropriate units for expressing this measurement. They used the concept of ratio to convert photographic measurement to real-life dimensions, demonstrating use of epistemological plane components as they applied conceptual knowledge, uses of dynamic geometry software, and length measurement units.

The following discussion occurred when they approximated the stadium area as 25,000 square meters:

PT19: How can it be 25 decare?

PT18: However, football field, would be approximately 10 decare.

PT17: Yes. A football field is approximately 10 decare. The stadium area could be 25 decare, which is a logical conclusion.

They determined the football field area by multiplying side lengths and stated that to find the real-life area equivalent, the measured area should be multiplied twice by the proportion constant. Their use of different representations and GeoGebra tools in these calculations further demonstrated use of representamen and artifact components of epistemological plane, while their reasoning about geometric properties and unit conversions reflected discursive genesis.

Although G5 reached a final result, no actions were observed in the interpretation and validation stages beyond unit conversion and plausibility checks. Their work primarily engaged instrumental and semiotic genesis during problem understanding and working mathematically, with discursive genesis emerging mainly during mathematization and reasoning about area conversions. Figure 7 shows G5's interactions between modeling phases and MWS.

Overall, G5's mathematical work demonstrated how transitions between real-world contexts and mathematical abstraction are mediated by different genesis processes. In the case of G5, components of the epistemological plane were primarily used during the understanding of the task and simplifying phases. The mathematization stage took place in the [Sem-Ins] vertical plane. The first part of the mathematically work phase involves

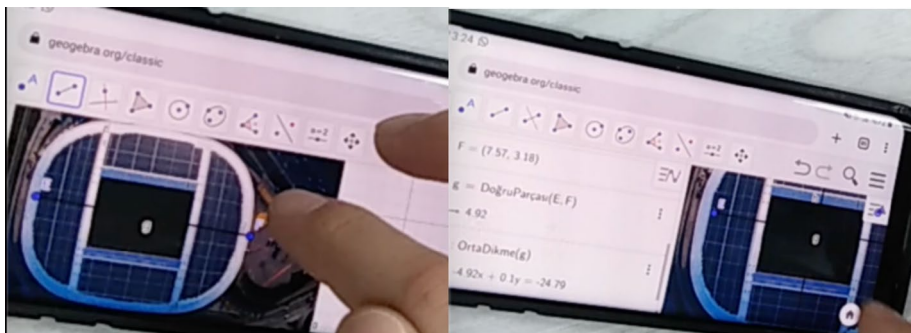


Fig. 6 Images of G5 during the mathematical work

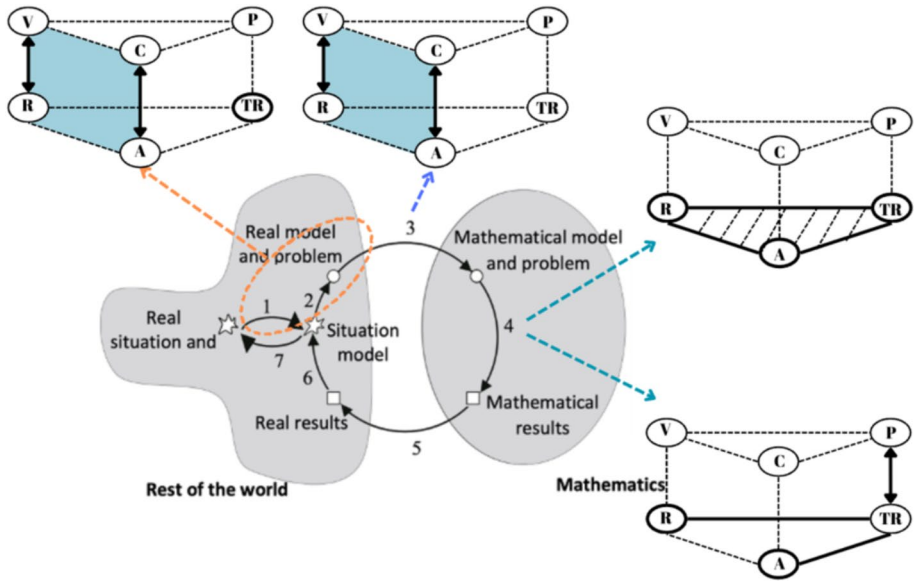


Fig. 7 Mathematical work of G5

epistemological plane’s components, whereas the second part reflects the emergence of semiotic genesis. No actions were observed in the interpretation and verification stages, although the mathematical work reached a final result. Thus, the analysis reveals preservice mathematics teachers in terms of which components are activated and which transitions are limited. However, the absence of deeper interpretation and validation stages suggests that their modeling proficiency was limited to construct and calculating, highlighting a need for further support in validating and generalizing mathematical results.

4.4 Mathematical working of the G7

G7 began by identifying the need for a bird’s-eye view of the stadium, which they retrieved online and imported into GeoGebra. They selected an automobile in the image as a reference object, created a line segment to represent it using the line segment tool, and measured its length with the algebra window. They noted that the stadium resembled an ellipse and created an ellipse around it using GeoGebra’s ellipse construction tool (Fig. 8). These actions demonstrated instrumental genesis, as they used technological tools to construct geometric models, and semiotic genesis, as they created and interpreted visual representations of the stadium and reference object. Furthermore, their consideration of the properties of ellipses reflected use of theoretical references.

After determining the length of the reference object, G7 proposed calculating the area of the solar panels by subtracting the football field area (approximated as a rectangle) from the stadium area at mathematization stage. They calculated the ratio between the measured length in the photograph and its real-life equivalent, representing discursive genesis as they used conceptual knowledge to plan the solution. Their construction of line segments and ellipse models in GeoGebra continued to illustrate instrumental

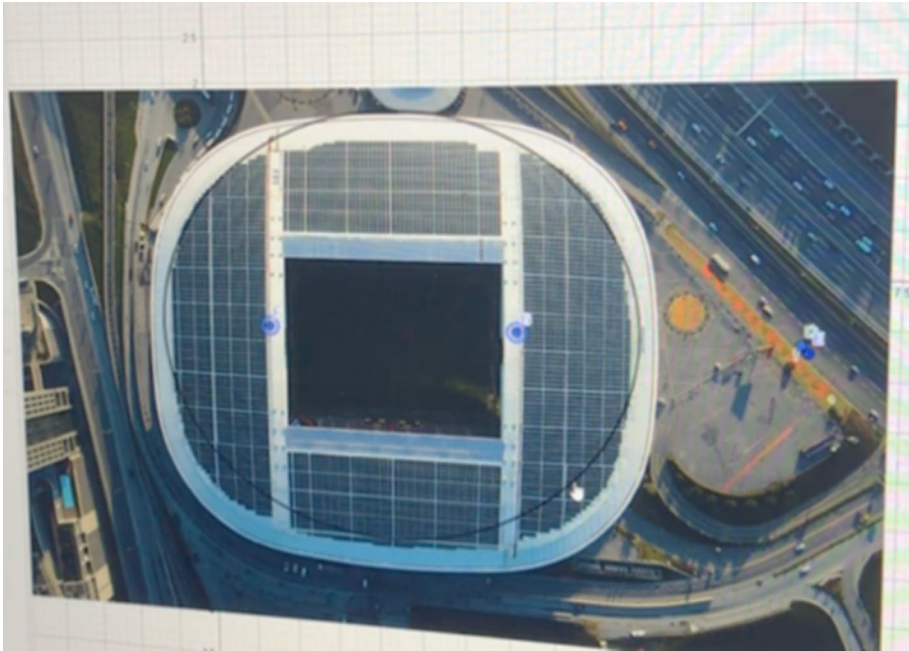


Fig. 8 Utilization of ellipse construction tool at GeoGebra

genesis. Conversely, calculating the ratio using the length measurement of the reference object in both the photograph and the real environment and formulating a solution plan on the basis of theoretical underpinnings of the ratio concept was deemed discursive genesis.

To solve the mathematical model at the working mathematically phase, the preservice teachers located the ellipse's center by constructing a line segment between its focal points and applying the midpoint tool, then constructed the radii by connecting the center to the ellipse boundary. These actions showed instrumental genesis, as they used GeoGebra tools for geometric construction, and semiotic genesis, as they represented and visualized the dimensions of the ellipse within their model.

Then, they initially assumed proportionality between area and length but realized this could introduce errors. To correct their approach, they placed points on the corners of the football field using the point tool, connected them with line segments, and measured their lengths with the algebra window. Here, discursive genesis emerged through reasoning about ratios and correcting misconceptions, while instrumental genesis was evident in their use of GeoGebra tools, and semiotic genesis in their creation and adjustment of geometric representations.

To verify their results, G7 searched online for average football field dimensions and compared them to their calculated values. They concluded that the results were close, indicating a successful approximation. This stage demonstrated discursive genesis as they justified results using external data and the combined epistemological plane components during

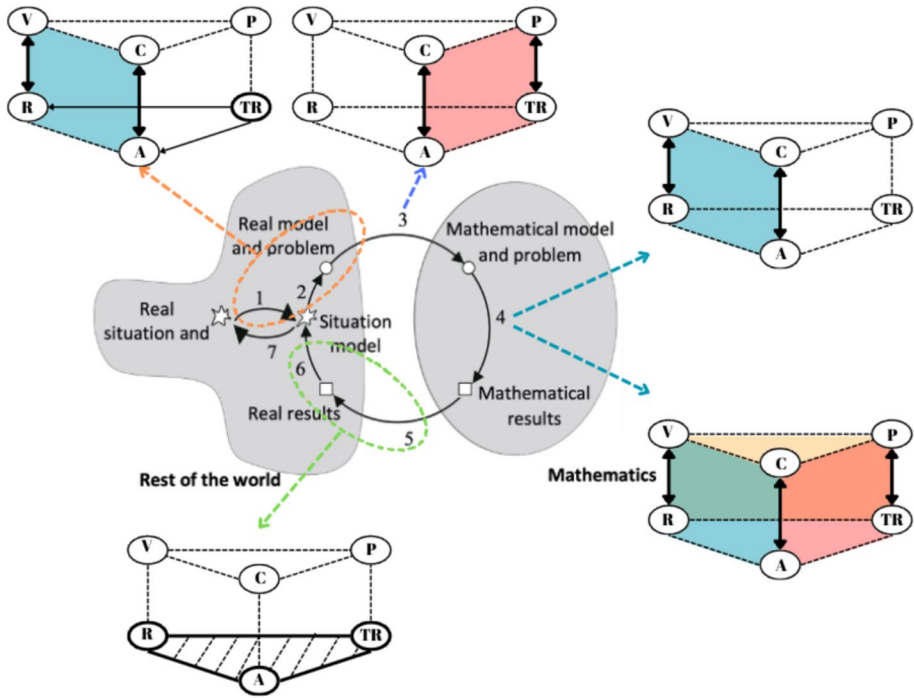


Fig. 9 Mathematical work of G7

their interpretation and validation processes. Figure 9 shows G7’s interactions between modeling phases and MWS.

Overall, G7’s mathematical work showed effective use of different genesis processes at the different phases. G7’s mathematical work initially took place in the [Sem-Ins] plane during the understanding of the task and simplification phases. The mathematization phase was situated within the [Sem-Dis] plane, with theoretical references supporting the use of artifacts and representations. The working mathematical phase proceeded in two steps: the first on the [Sem-Ins] plane and the second involving all the components of the MWS. The interpretation and verification phases reflected the use of epistemological plane components. Their modeling process engaged all vertical planes of the MWS and included successful interpretation and validation stages, indicating a high level of modeling proficiency with strong integration of cognitive and epistemological plane components.

5 Conclusion and discussion

This study examined preservice teachers’ actions while solving a mathematical modeling task within the MWS framework. The analysis demonstrated that MWS provides a powerful lens for understanding how epistemological components and cognitive processes interact through modeling. Findings revealed that instrumental, semiotic, and discursive genesis collectively mediate transitions between real-world contexts and mathematical abstractions, highlighting their crucial role in preservice teachers’ modeling competencies.

In this study, groups scoring below 7 on the MCAR were excluded, as such indicate no modeling competence per the rubric. Analysis showed that while these preservice teachers attempted to identify given data and required quantities during understanding and simplification, they struggled to activate epistemological plane components such as representations and artifacts. Consequently, they failed to generate meaningful connections and experienced difficulties during mathematization and working mathematically stages, where genesis processes were not adequately realized. They were also unable to justify area measurement concepts, avoided using technological tools as functional artifacts, and relied on intuition rather than appropriate mathematical representations. These limitations hindered their engagement with the modeling cycle and highlighted the need to integrate epistemological components and genesis processes into preservice teacher education for modeling tasks.

The analysis of G1 revealed that their modeling process did not progress linearly through the modeling cycle. While they effectively utilized epistemological plane components during understanding and simplification stages, their initial mathematization was insufficient due to ineffective actions during the working mathematically phase, requiring them to revisit and repeat mathematization. Specifically, although their work on [Sem-Ins] plane and in the working mathematically phase did not resolve the problem, subsequent actions on the [Sem-Dis] plane enabled them to mathematize the task effectively. Ultimately, a mathematical result was achieved through actions performed on the [Ins-Dis] plane, where artifacts were integrated with theoretical justifications. This finding aligns with Reyes Avendaño (2020), who noted that modeling processes often involve iterative rather than linear progression.

In contrast, G2 and G7 demonstrated linear modeling processes, engaged all three vertical planes during the working mathematically phase, and effectively integrated representations, artifacts, and theoretical reasoning. In the understanding and simplification stages, they acted on the [Sem-Ins] plane, coordinating visual representations with technological tools. During mathematization, G2's semiotic genesis actions on the [Sem-Dis] plane showed connections between representations and reasoning, while G7 integrated theoretical references with artifact use, demonstrating discursive and instrumental genesis. Furthermore, both groups included interpretation and validation stages, with G2 focusing on transforming representations and reasoning within the [Sem-Dis] plane, and G7 validating results using components of the epistemological plane. These findings indicate that G2 and G7 engaged in deeper and more comprehensive mathematical work than G1 and G5, emphasizing the importance of activating epistemological components early to support successful modeling.

Overall, examination of the mathematical works showed that actions during understanding, simplification, and mathematization stages shared similar characteristics. In the understanding and simplification, preservice teachers consistently used measurable quantities of a reference object to establish scale and connect real-world context to mathematical representations. During the mathematization stage, they modeled solutions with geometric shapes of known area formulas. These findings highlight that activating epistemological plane components early supports subsequent modeling processes. Moreover, activating genesis processes early in the modeling cycle facilitates solution-oriented mathematical work by enabling systematic visualization, construction, and reasoning. For instance, in G1, semiotic genesis, involving creation and transformation of representations, co-occurred with discursive genesis during the working mathematically phase. This aligns with Vergo Hernández et al. (2023), who concluded that semiotic genesis related to task concepts is critical for the progression of modeling processes. Thus, this study extends these prior

insights by demonstrating MWS's explanatory power in revealing how genesis processes structure and advance preservice teachers' modeling work.

A comparison of the mathematical models developed by preservice teachers revealed high similarity across groups. Participants consistently began by obtaining a bird's-eye view photograph, considering perspective distortion, and identifying a reference object to establish scale. They detected geometric shapes within the image and used known formulas to approximate areas. Because of the perspective in such images, their results were close estimations based on scaling with the reference object. This aligns with Schukajlow (2013), who similarly found that tasks involving photographs with perspective introduce interpretive and representational challenges requiring careful coordination of visual information and mathematical reasoning. Furthermore, groups that struggled to activate genesis processes effectively during the working mathematically phase often failed to reach interpretation and validation stages. These findings suggest that engaging in diverse actions and activating multiple genesis processes is crucial for interpretation and validation. This supports Henríquez Rivas et al. (2023), who emphasized that limiting mathematical work to a single vertical plane compromises the depth and completeness of the modeling.

The analysis of the interpretation and validation stages revealed that preservice teachers who did not activate genesis processes effectively during the initial stages of the modeling stages struggled to enrich and extend their mathematical work. These participants often could not engage in interpretation and validation, limiting the depth of their outcomes. This finding aligns with Jablonski (2023), who noted that few mathematical works incorporated these stages effectively. In contrast, preservice teachers who supported early modeling stages with various genesis processes could activate additional genesis processes later, producing more comprehensive and robust mathematical work. Furthermore, those who fully completed the working mathematically phase also engaged in diverse interpretation and validation actions, using components from both epistemological and vertical planes of MWS. These observations highlight the critical importance of engaging genesis processes early to enable meaningful interpretation, validation, and generalization of results.

These findings also have practical implications for preservice mathematics teacher education. Designing modeling activities that deliberately mobilize epistemological components and multiple genesis processes can help future teachers move beyond procedural calculation toward interpretation, validation, and generalization. Integrating the MWS into teacher preparation may also raise awareness of the cognitive and epistemological demands of modeling, equipping teachers with tools to design and orchestrate tasks that foster complete mathematical work.

This study has certain limitations. The small sample and focus on a single modeling task restrict generalizability, while group-based data collection and reliance on video and written reports may have obscured individual reasoning and cognitive processes. These factors should be considered when interpreting the results. Future research should involve larger and more diverse samples, varied modeling contexts, and mixed methods to strengthen external validity. In addition, examining conditions that facilitate or hinder preservice teachers' mathematical work, comparing genesis processes across tasks with different representations and digital tools, and combining qualitative analyses with quantitative approaches such as pre-post assessments or intervention studies could provide deeper insights into how MWS-based modeling competencies develop over time and across contexts.

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Author contribution All authors contributed to the conceptual design, data analysis, and final review of the manuscript.

Data availability The data that support the findings of this study are available from the corresponding author upon request.

Code availability The transcripts are coded with MAXQDA24.

Declarations

Ethics approval This study was conducted in accordance with the Academic Integrity Code of Balıkesir University.

Consent to participate The consent of the participants was obtained with an informed consent form.

Consent for publication This study was conducted with the permission of the Dean's Office of Balıkesir University Necatibey Faculty of Education.

Competing interests The authors declare no competing interests.

References

- Balıkesir University. (2022). *Necatibey Eğitim Fakültesi / İlköğretim Matematik Öğretmenliği – Dersler [Necatibey Faculty of Education / Elementary Mathematics Teaching – Courses]*. Retrieved May 14, 2022, from <https://www.balikesir.edu.tr/>
- Blum, W., & Leiß, D. (2007). How do students' and teachers deal with modelling problems? In P. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 222–231). Horwood. <https://doi.org/10.1533/9780857099419.5.221>
- Bukova Güzel, E. (Ed.). (2016). *Matematik eğitiminde matematiksel modelleme: Araştırmacılar, eğitimciler ve öğrenciler için* [Mathematical modelling in mathematics education: For researchers, educators and students]. Pegem Akademi. <https://doi.org/10.14527/9786053185444>
- Büyükköztürk, Ş., Kılıç-Çakmak, E., Akgün, Ö., Karadeniz, Ş., & Demirel, F. (2008). *Bilimsel araştırma yöntemleri* [Scientific research methods]. Pegem Akademi.
- Chamberlin, S. A., & Moon, S. M. (2005). Model-eliciting activities as a tool to develop and identify creatively gifted mathematicians. *Journal of Secondary Gifted Education, 17*(1), 37–47. <https://doi.org/10.4219/jsge-2005-393>
- Derouet, C., Kuzniak, A., Nechache, A., Parzys, B., & Vivier, L. (2017). The mathematical working space model: An open and adaptable theoretical framework? CERME 10.
- Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie: De développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales De Didactique Et Sciences Cognitives, 10*, 5–53.
- Ekol, G., & Greenop, S. (2023). Teacher interventions using guided discovery and mathematical modelling in grade 10 financial mathematics. *The European Educational Researcher, 6*(2), 35–53. <https://doi.org/10.31757/euer.623>
- Erlandson, D. A., Harris, E. L., Skipper, B. L., & Allen, S. D. (1993). *Doing naturalistic inquiry: A guide to methods*. Sage.
- Espinoza-Vásquez, G., Henríquez-Rivas, C., Climent, N., Ponce, R., & Verdugo-Hernández, P. (2024). Teaching thales's theorem: Relations between suitable mathematical working spaces and specialised knowledge. *Educational Studies in Mathematics, 118*, 271–293. <https://doi.org/10.1007/s10649-024-10367-9>
- Enerjisa Energy. (n.d.). Galatasaray. Retrieved September 17, 2022, from <https://www.enerjisa.com.tr/tr/isimin-enerjisi/referanslar/projelerimiz/galatasaray>
- Galatasaray Sports Club. (n.d.). Ali Sami Yen Sports Complex Nef Stadium to meet its electricity needs with solar energy. Retrieved September 17, 2022, from <https://www.galatasaray.org/haber/kulup/enerjisa-ve-galatasaray-dan-bir-dunya-rekoru/50696#:~:text=Stadyumun%20çatısına%20kurulan%2010%20bini,olarak%204.650%20MWh%20elektrik%20üretecek>

- Henríquez Rivas, C., Ponce, R., Carillo Yáñez, J., Climent, N., & Espinoza-Vásquez, G. (2021). Trabajo matemático de un profesor basado en tareas y ejemplos propuestos para la enseñanza [Mathematical work by a teacher based on tasks and examples proposed for teaching]. *Enseñanza De Las Ciencias*, 39(2), 123–142. <https://doi.org/10.5565/rev/ensciencias.3210>
- Henríquez Rivas, C., Verdugo Hernández, P., & Valenzuela Barrera, Y. (2023). Trabajo matemático de estudiantes de educación técnica profesional en un contexto interdisciplinar [Mathematical work of vocational technical education students in an interdisciplinary context]. *Uniciencia*, 37(1), 1–24. <https://doi.org/10.15359/ru.37-1.25>
- Herget, W., & Torres-Skoumal, M. (2007). Picture (im) perfect mathematics! In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 379–386). Springer. https://doi.org/10.1007/978-0-387-29822-1_41
- Jablonski, S. (2023). Is it all about the setting? — A comparison of mathematical modelling with real objects and their representation. *Educational Studies in Mathematics*, 113(2), 307–330. <https://doi.org/10.1007/s10649-023-10215-2>
- Kotze, H. (2017). Towards contexts with mathematical modelling tasks. In J. Kriek, B. effective, K. Padayachee, S. van Putten, H. Atagana, H. Letseka, W. Rauscher, & S. Faleye (Eds.), *Towards effective teaching and meaningful learning in mathematics, science and technology education* (pp. 192–198). UNISA.
- Kuzniak, A. (2022). The theory of mathematical working spaces—Theoretical characteristics. In *Mathematical work in educational context* (pp. 3–31). https://doi.org/10.1007/978-3-030-90850-8_1
- Kuzniak, A., & Nechache, A. (2021). On forms of geometric work: A study with preservice teachers based on the theory of mathematical working spaces. *Educational Studies in Mathematics*, 106(2), 271–289. <https://doi.org/10.1007/s10649-020-10011-2>
- Kuzniak, A., Nechache, A., & Drouhard, J. P. (2016a). Understanding the development of mathematical work in the context of the classroom. *ZDM – Mathematics Education*, 48(6), 861–874. <https://doi.org/10.1007/s11858-016-0773-0>
- Kuzniak, A., Nechache, A., & Richard, P. R. (2022). The theory of mathematical working spaces in brief. In A. Kuzniak, E. Montoya-Delgadillo, & P. R. Richard (Eds.), *Mathematical work in educational context: The perspective of the theory of mathematical working spaces* (pp. 57–69). Springer International Publishing. https://doi.org/10.1007/978-3-030-90850-8_3
- Kuzniak, A., & Richard, P. R. (2014). Spaces for mathematical work: Viewpoints and perspectives. *Revista Latinoamericana de Investigación en Matemática Educativa*, 17(4–1), 17–27. <https://doi.org/10.12802/relime.13.1741b>
- Kuzniak, A., Tanguay, D., & Elia, I. (2016b). Mathematical working spaces in schooling: An introduction. *ZDM – Mathematics Education*, 48(6), 721–737. <https://doi.org/10.1007/s11858-016-0812-x>
- Lagrange, J.-B., Huincahue, J., & Psycharis, G. (2022). Modeling in education: New perspectives opened by the theory of mathematical working spaces. In A. Kuzniak, E. Montoya-Delgadillo, & P. R. Richard (Eds.), *Mathematical work in educational context: The perspective of the theory of mathematical working spaces* (pp. 247–266). Springer International Publishing. https://doi.org/10.1007/978-3-030-90850-8_11
- Ledezma, C., Font, V., & Sala, G. (2023). Analysing the mathematical activity in a modelling process from the cognitive and onto-semiotic perspectives. *Mathematics Education Research Journal*, 35(4), 715–741. <https://doi.org/10.1007/s13394-022-00411-3>
- Lesh, R., & Doerr, H. (2003). Foundations of a models and modeling perspective on mathematics teaching and learning. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on mathematics teaching, learning, and problem solving*. Lawrence Erlbaum. <https://doi.org/10.4324/9781410607713>
- Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. Sage. [https://doi.org/10.1016/s1098-2140\(99\)80125-8](https://doi.org/10.1016/s1098-2140(99)80125-8)
- MoNE. (2018). *Matematik Dersi Öğretim Programı* [Mathematics course curriculum]. Ankara
- MoNE. (2024). *Türkiye Yüzyılı Maarif Modeli Öğretim Programları Ortak Metni* [Century of Türkiye education model curriculum common text]. Ankara.
- Nechache, A., & Gómez-Chacon, I. M. (2022). Methodological aspects in the theory of mathematical working spaces. In A. Kuzniak, E. M. Delgadillo, & P. R. Richard (Eds.), *Mathematical work in educational context: The perspective of the theory of mathematical working spaces*. Springer. <https://doi.org/10.1007/978-3-030-90850-8>
- Reyes Avendaño, C. G. (2020). *Enseignement et apprentissage des fonctions numériques dans un contexte de modélisation et de travail mathématique* [Teaching and learning numerical functions in the context of modeling and mathematical work]. [Doctoral dissertation, Université Paris Cité]. Paris.

- Schukajlow, S. (2013). Lesekompetenz und mathematisches modellieren [Reading skills and mathematical modeling]. In R. Borromeo Ferri, G. Greefrath, & G. Kaiser (Eds.), *Mathematisches Modellieren für Schule und Hochschule: Theoretische und didaktische Hintergründe* (pp. 125–143). Springer Fachmedien Wiesbaden. https://doi.org/10.1007/978-3-658-01580-0_6
- Stillman, G. A., Kaiser, G., Blum, W., & Brown, J. P. (2013). Mathematical modelling: Connecting to teaching and research practices—The impact of globalisation. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 1–24). Dordrecht: Springer. https://doi.org/10.1007/978-94-007-6540-5_1
- Tekin Dede, A. (2015). *Matematik derslerinde öğrencilerin modelleme yeterliklerinin geliştirilmesi: bir eylem araştırması* [Developing students' modeling competencies in mathematics courses: An action research]. (No: 395238) [Doctoral dissertation, Dokuz Eylül University]. YÖK National Thesis Center. <https://tez.yok.gov.tr/UlusalTezMerkezi/TezGoster?key=WBc656i315e2eV6-EZV1ohvx1xrSAXczILJabrTSUPqmxbtn5Td3FSjgJvvtTsg>
- Tekin Dede, A., & Bukova Güzel, E. (2018). A rubric development study for the assessment of modeling skills. *The Mathematics Educator*, 27(2), 33–72. <https://doi.org/10.63301/tme.v27i2.2040>
- Verdugo Hernández, P., Huincahue, J., Cumsille, P., & Nechache, A. (2023). Articulación entre modelización y espacios de trabajo matemático. Análisis de una tarea bajo el ciclo de Blomhøj en educación superior [Articulation between modeling and mathematical working spaces. Analysis of a task under the Blomhøj cycle in higher education]. *Acta Scientiae*, 24(7), 146–175. <https://doi.org/10.17648/acta.scientiae.7135>
- Yıldırım, A., & Şimşek, H. (2021). *Sosyal Bilimlerde Nitel Araştırma Yöntemleri* [Qualitative research methods in social sciences]. Seçkin Publishing.

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