RESEARCH ARTICLE

Lattice-Based Decision Models for Green Urban Development: Insights from L_q* q-Rung Orthopair Multi-fuzzy Soft Set

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Abstract

Location selection is a critical process in decision-making for projects that involve multiple criteria, such as urban planning, industrial site development, or green building projects. Multiple criteria decision making (MCDM) is a systematic approach that evaluates and ranks potential alternatives based on a set of often conflicting criteria. This study focuses on selecting the optimal urban location for a green building project by employing the L_q* q-rung orthopair multi-fuzzy soft-MCDM(L_q* q-ROMFS) techniques. The L_q* q-ROMFS set combines elements from two distinct theories with lattice ordering parameters: q-rung orthopair fuzzy set and multi-fuzzy soft set. It provides a mathematical framework with multiple parameters that effectively represents problems involving multi-dimensional data within a dataset. We expand this concept by establishing the algebraic structures of L_q* q-ROMFS sets, including properties like modularity and distributivity, while also analyzing their homomorphism under lattice mappings. Finally, leveraging the L_q* q-ROMFS matrix, we propose both a choice matrix and a weighted choice matrix to effectively address the selection of the optimal urban location for a green building project.

Keywords L_q* q-ROMFS-MCDM · Green building project · Modularity · Distributivity · Homomorphism · Choice matrix

Abbreviations

MCDM	Multi-criteria decision-making
MDs	Membership degree
NMDs	Non-membership degree
FSs	Fuzzy sets
MF	Sets-multi-fuzzy sets
IFS	Intuitionistic fuzzy Set
PyFSs	Pythagorean fuzzy set
FFSs	Fermatean fuzzy set
q-ROMFSs	q-rung orthopair fuzzy set
SSs	Soft sets
IFSS	Intuitionistic fuzzy soft set
PyFSS	Pythagorean fuzzy soft sets
IMFSS	Intuitionistic fuzzy soft Set
q-ROMFSS	q-rung orthopair multi-fuzzy soft set

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LDMFSS	Linear diophantine multi-fuzzy soft set
L_q^* q-ROMFSS	L_q^* q-rung orthopair multi-fuzzy sets

1 Introduction

Identifying the ideal urban location for a green building project is a prime example of a complex MCDM task, requiring the evaluation of multiple criteria, such as environmental impact, accessibility, cost, and sustainability, all under conditions of uncertainty. Multi-Criteria Decision-Making (MCDM) involves ranking or selecting alternatives based on multiple criteria, often in uncertain conditions. In complex MCDM scenarios, enterprises must adapt quickly to changing customer needs. A systematic approach evaluates conflicting criteria effectively, enabling informed decisions. Zadeh [61] introduced Fuzzy Sets, using Membership Degrees (MDs) from a unit interval to address uncertainty in decision-making.

Multi-fuzzy sets, an advanced extension of fuzzy set theory, effectively tackle challenges like pixel color analysis and image recognition, which traditional methods struggle to address. The intuitionistic fuzzy set (IFS) extends fuzzy sets (FS) by incorporating membership, non-membership (NMDs), and hesitation degrees (HDs), offering greater flexibility in handling fuzziness and uncertainty. To address limitations in FS and IFS frameworks, Pythagorean fuzzy sets (PyFSs) were introduced, allowing broader representation of uncertainty. However, PyFSs faced constraints, leading to the development of Fermatean fuzzy sets (FFSs), which handle more complex uncertainty cases. Later, q-rung orthopair fuzzy sets (q-ROFSs) generalized these concepts, with their flexibility covering IFS, PFS, and FFS as special cases depending on the value of q.

Soft sets (SS) were initially proposed to address decision-making issues involving multiple options, and their applications in decision-making have been further developed. This progress led to the creation of fuzzy soft sets (FSS) and intuitionistic fuzzy soft sets (IFSS), which were eventually extended to Pythagorean fuzzy soft sets (PyFSS). By merging multi-fuzzy sets and soft set models, the concept of multi-fuzzy soft sets was introduced and applied to complex decision-making scenarios. To better handle decision-making problems, the intuitionistic multi-fuzzy soft set (IMFSS) was developed. Recent advancements include the q-rung orthopair multi-fuzzy soft set (q-ROMFSS) and linear diophantine multi-fuzzy soft set (LDMFSS), which provide enhanced decision-making tools.

Lattice theory plays a crucial role in various fields of everyday life. However, the lack of lattice-ordered frameworks in hybrid sets led to further advancements to expand their applicability. Lattice-ordered MFSS was introduced and applied in multiple contexts. Recently, L_q* q-rung orthopair multi-fuzzy soft sets(L_q* q-ROMFS sets), an extension of lattice-ordered MFSS, was developed by integrating multi-NMDS. Additionally, lattice-ordered linear diophantine MFSS was proposed and applied in myocardial infarction prognostication.

1.1 Literature Review

Several mathematical tools have been developed to tackle real-world uncertainties. Notably, fuzzy sets(FSs), introduced by Zadeh [61], handle imprecise information through partial membership of elements, which is useful in scenarios where binary evaluations fall short. Atanassov [9] introduced intuitionistic fuzzy sets (IFSs) as a generalization of FSs, integrating both MDs and NMDs, with their sum bounded by unity. In 2014, Yager [58] expanded the concept of intuitionistic fuzzy sets to Pythagorean fuzzy sets (PFS), providing a broader framework to aid in decision-making. Later, Yager [57, 59] proposed the concept of generalized orthopair fuzzy sets, which extend the IFSs and the PFSs. This extension allows for a broader representation of data vagueness by applying a qth power constraint, meaning that the sum of the qth powers of the MDs and NMDs remains within the limit of one.

In 1999, Molodtsov [36] introduced the innovative concept of soft set theory, providing a fresh approach to modeling vagueness and uncertainty. Research on soft sets is progressing rapidly in the field of uncertainty studies.

For instance, Maji et al. [34, 35] introduced fuzzy soft sets / intuitionistic fuzzy soft sets as a fuzzy generalization of soft sets and explored their applications in decision-making processes. Peng et al. [39] studied Pythagorean fuzzy soft sets, their binary operations, and proposed a decision-making algorithm. Hussain et al. [21] initiated the idea of q-rung orthopair fuzzy soft sets by combining the theories of soft set and q-rung orthopair fuzzy set. Recent developments in MCDM include the introduction of q-rung orthopair fuzzy sets [22, 48, 49] and interval-valued fuzzy approaches [24, 46, 47, 50], which extend classical methods to handle higher levels of uncertainty and vagueness in criteria evaluation.

Sebastian and Ramakrishnan first introduced the concept of multi-fuzziness in 2011 [51, 52]. Yang et al. [60] then combined multi-fuzzy sets with soft sets to create a new mathematical model called the multi-fuzzy soft set. Later, Dey and Pal [19] extended the concept of MFSS, while Zhang and Shu [62] further developed it by introducing possibility MFSS and applying it to MCDM. Subsequently, Das and Kar defined the concept of the intuitionistic multi-fuzzy soft set in [17]. In 2018, Al-Qudah et al. [1] introduced CMFSS, which integrates the advantages of both CMFS and soft sets. Recently, the concept of q-ROMFS sets [53] has been aimed at modeling uncertainties in a multi-dimensional context. This approach accommodates q-ROMF values for both alternatives and parameters. Various algebraic structures and their applications have been explored within the context of multi-fuzzy soft sets [1, 5, 18].

Birkhoff's pioneering work in 1930 laid the foundation for the general development of lattice theory [14], which has since found applications in numerous fields. Recently, efforts have been made to integrate soft set theory with lattice theory and fuzzy set theory. Karaaslan and Cagman [29, 30] introduced the concept of soft lattices and fuzzy soft lattices along with their fundamental properties. They also explored the relationships between different types of soft lattices, providing a comprehensive understanding of their structure and interconnections. Additionally, Karaaslan et al. [30] developed the theory of fuzzy soft lattices and examined various related properties. Nagarajan and Geetha [37] defined duality in soft lattices and examined properties of modular and distributive soft lattices.

The concept of a lattice-ordered fuzzy soft set (LOFSS), introduced in [7], merges lattice-ordered soft sets [2] with fuzzy sets, offering an effective method for ranking parameters in decision-making and providing significant advantages. Mahmood et al. [33] and Muhammad Bilal Khan et al. [31] introduced lattice-ordered IFSS and lattice (anti-lattice) ordered double-framed soft sets (LODFrSS). Sabeena Begam et al. [12] developed and applied lattice-ordered MFSS (LOMFSS) in various fields. Due to the limitations of existing lattice-ordered frameworks, Vimala et al. proposed various theories on lattice ordering structures [23, 54, 55] and expanded their applicability to diverse domains. A recent advancement involves extending LOMFSS by incorporating multi-NMDs [40, 41]. For further exploration of related studies, refer to the provided references [5, 18, 26]. Several studies have explored lattice algebraic structures and their applications in multi-fuzzy soft sets, who have contributed to understanding lattice-ordered multi-fuzzy soft sets and their algebraic properties ([4, 11, 13, 28]).

Decision-making in the selection of construction methods for green buildings involves complex criteria that necessitate advanced analytical tools [10, 20, 38, 44, 45]. Recent developments in MCDM [15, 16] include the introduction of interval valued q-rung orthopair fuzzy soft sets [63], Pythagorean fuzzy sets [3] and Complex fuzzy approaches [6, 8], which extend classical methods to handle higher levels of uncertainty and vagueness in criteria evaluation. Incorporating uncertainty into decision-making, q-rung orthopair fuzzy MCDM methods [25, 27, 32] have provided significant advancements by effectively modeling imprecise and subjective judgments. Recent studies have explored various fuzzy operators for MCDM [42, 43, 56]. Retaining the benefits of q-ROMFS sets and emphasizing the significance of lattice ordering the parameters, this article establishes the theoretical frameworks of L_q* q-ROMFS sets. These enhancements increase the flexibility and precision of decision-making processes, allowing for more accurate modeling and analysis in complex situations where data uncertainty and vagueness are prevalent.

1.2 Main Contribution

- To enhance the theoretical foundation of L_q^* q-ROMFS sets by developing their algebraic structures, including modularity, distributivity, and homomorphism between two L_q^* q-ROMFS sets, along with illustrative examples.
- To introduce the concepts of a choice matrix and a weighted choice matrix for L_q^* q-ROMFS sets, accommodating both equal and unequal weights, respectively.
- A systematic MCDM approach is developed using the proposed L_q^* q-ROMFS set framework. This methodology incorporates multi-membership and multi-non-membership values for each criterion while employing lattice ordering to structure the criteria.

1.3 Research Gap and Motivation

- While many MCDM techniques exist for fuzzy sets, there is a lack of approaches integrating multi-membership and multi-non-membership values with lattice ordering in the q-ROMFS framework. This research addresses this gap by proposing an MCDM technique tailored for q-ROMFS sets with lattice ordering, enabling more nuanced decision-making in uncertain scenarios.
- Existing methods face challenges in effectively managing multi-dimensional fuzzy data with lattice ordering parameters in MCDM scenarios, particularly in practical applications such as urban planning, green energy initiatives, and industrial development.
- Several unique properties are introduced by incorporating algebraic structures into L_q* q-ROMFS sets, significantly improving their ability to handle uncertainty and imprecision.

The remainder of this article is organized into five sections. Section 2 presents foundational definitions crucial for developing the proposed framework. Section 3 explores the algebraic structures of L_q^* q-ROMFS sets. Section 4 outlines a novel MCDM algorithm utilizing choice and weighted choice matrices of L_q^* q-ROMFS sets, accompanied by a case study on selecting the best urban location for a green building project. Section 5 provides a robustness analysis, with the final section offering concluding remarks. Figure 1 illustrates the flowchart of the proposed approach.

2 Preliminaries

Definition 2.1 [14] A partial order $\leq$ on the set $(L_Y, \leq)$ is a lattice if the set $L_{\breve{\mathfrak{z}}} = \{l_1, l_2\}$ has a supremum and an infimum in $L_{\breve{\mathfrak{z}}}$, $\forall\, l_1, l_2 \in L_{\breve{\mathfrak{z}}}$. If $\exists$ 0,1 $\in L_{\breve{\mathfrak{z}}} \ni 0 \leq$ u ≤ 1, $\forall$ u $\in L_{\breve{\mathfrak{z}}}$, then $L_{\breve{\mathfrak{z}}}$ is a bounded lattice.

Definition 2.2 [53] A q-rung orthopair multi-fuzzy set (q-ROMF set) $\breve{\mathfrak{M}}$ of dimension m over $\breve{\mathfrak{Z}}$ is a set of ordered sequences of the form:

$$\breve{\mathfrak{M}} = \left\{ \left\langle \breve{\mathfrak{z}}, \left(\mu^1_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}}), \nu^1_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}})\right), \left(\mu^2_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}}), \nu^2_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}})\right), \cdots, \left(\mu^r_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}}), \nu^r_{\breve{\mathfrak{M}}}(\breve{\mathfrak{z}})\right) \right\rangle : \breve{\mathfrak{z}} \in \breve{\mathfrak{Z}} \right\}$$

The functions $\mu^m_{\breve{\mathfrak{M}}}$ and $\nu^m_{\breve{\mathfrak{M}}}: \breve{\mathfrak{z}} \to [0, 1]$ for m= $1, 2, \cdots$, r indicates the degree of membership and non-membership in a multi-dimensional context, respectively with the restriction that $0 \leq (\mu^m_{\breve{\mathfrak{M}}})^q + (\nu^m_{\breve{\mathfrak{M}}})^q \leq 1$. The collection of q-ROMF set of dimension m over $\breve{\mathfrak{Z}}$ is denoted as q-ROMFS$^{\breve{\mathfrak{z}}}$.

Definition 2.3 [53] A q-ROMFS set (q-ROMFS set) of dimension m over $\breve{\mathfrak{Z}}$ is defined as $(\breve{F}, \Delta)$ where $\breve{F}: \Delta \to q - ROMFS^{\breve{\mathfrak{z}}}$ and $\Delta \subseteq \mathfrak{B}$ (the parameter set). It is represented as follows:

$$(\breve{F}, \Delta) = \{\langle \rho, \breve{F}(\rho) \rangle, \rho \in \Delta, \breve{F}(\rho) \in q - ROMFS^{\breve{\mathfrak{z}}}\}$$

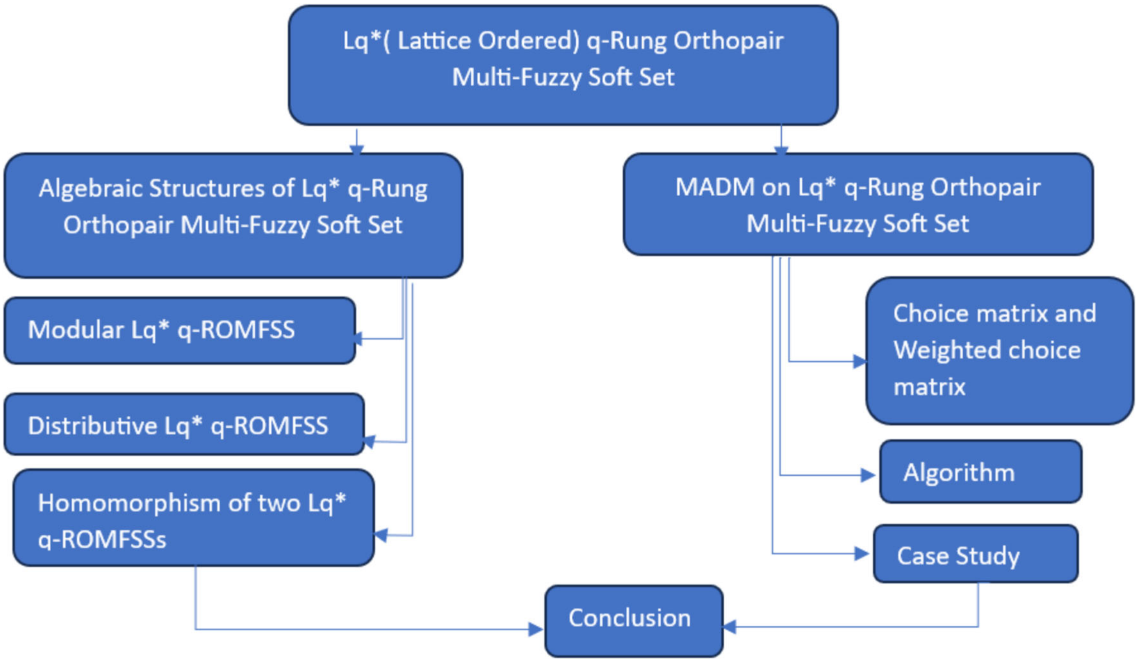

Fig. 1 Flowchart of the proposed approach

$$where\ \breve{F}(\rho) = \{\langle \breve{\mathfrak{z}}, \mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}})\rangle : \breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}, m = 1, 2, \ldots r\ \ and\ \ q \geq 1\}$$

Definition 2.4 [40] A q-ROMFS set $(\breve{F}, \Delta)$ of dimension m over $\breve{\mathfrak{Z}}$ is defined as L_q*(lattice ordered) q-ROMFS set (L_q* q-ROMFSS) if for $\rho_1, \rho_2 \in \Delta$ such that $\rho_1 \geq \rho_2$ implies $\breve{F}(\rho_1) \breve{\subseteq} \breve{F}(\rho_2)$ i.e.,

$$\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \leq \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\ and\ \nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \geq \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),$$

for all $\breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}$ and $m = 1, 2, \cdots r$.
Then we define a matrix is as follows:

$$[\mathcal{H}] = [h_{ij}] = [(\mu^m_{\breve{F}_{ij}}, \nu^m_{\breve{F}_{ij}})]_{s\times t}$$
$$= \begin{pmatrix} [(\mu^1_{\breve{F}_{11}}, \nu^1_{\breve{F}_{11}}), \ldots, (\mu^r_{\breve{F}_{11}}, \nu^r_{\breve{F}_{11}})] \cdots [(\mu^1_{\breve{F}_{1t}}, \nu^1_{\breve{F}_{1t}}), \ldots, (\mu^r_{\breve{F}_{1t}}, \nu^r_{\breve{F}_{1t}})] \\ [(\mu^1_{\breve{F}_{21}}, \nu^1_{\breve{F}_{21}}), \ldots, (\mu^r_{\breve{F}_{21}}, \nu^r_{\breve{F}_{21}})] \cdots [(\mu^1_{\breve{F}_{2t}}, \nu^1_{\breve{F}_{2t}}), \ldots, (\mu^r_{\breve{F}_{2t}}, \nu^r_{\breve{F}_{2t}})] \\ \vdots \qquad\qquad \ddots \qquad\qquad \vdots \\ [(\mu^1_{\breve{F}_{s1}}, \nu^1_{\breve{F}_{s1}}), \ldots, (\mu^r_{\breve{F}_{s1}}, \nu^r_{\breve{F}_{s1}})] \cdots [(\mu^1_{\breve{F}_{st}}, \nu^1_{\breve{F}_{st}}), \ldots, (\mu^r_{\breve{F}_{st}}, \nu^r_{\breve{F}_{st}})] \end{pmatrix}_{s\times t}$$

which is called a $L_q * q - ROMFS$ matrix of order s × t corresponding to the L_q* q-rung orthopair multi-fuzzy soft set $(\breve{F}, \Delta)$ of dimension m over $\breve{\mathfrak{Z}}$.

Example 2.5 In the global market, four software companies, $\breve{\mathfrak{Z}}_1, \breve{\mathfrak{Z}}_2, \breve{\mathfrak{Z}}_3$, and $\breve{\mathfrak{Z}}_4$, develop antivirus software solutions. Their most significant customer base is in North America. Aside from North America, these companies primarily target regions in Europe and Asia for advertising and product launches. It's worth noting that North America includes the United States and Canada. Therefore, it is desirable to evaluate the performance of each company's antivirus software in three regions: North America, Europe, and Asia. The lattice order is determined by the level of protection effectiveness, from lowest to highest.

Since each of these companies produces both consumer-grade and enterprise-grade antivirus solutions, the evaluation committee decided to assess each product type separately.The representation of $L_q * q - ROMFSS$ is shown below:

$$(\breve{F},\Delta)=\left\{\begin{array}{l}\left\{\breve{F}(\rho_1)=\left\{\frac{\breve{\mathfrak{Z}}_1}{[(0.3,0.7),(0.4,0.8)]},\frac{\breve{\mathfrak{Z}}_2}{[(0.5,0.7),(0.3,0.6)]},\frac{\breve{\mathfrak{Z}}_3}{[(0.3,0.8),(0.2,0.6)]},\frac{\breve{\mathfrak{Z}}_4}{[(0.4,0.5),(0.5,0.6)]}\right\}\right.\\ \breve{F}(\rho_2)=\left\{\frac{\breve{\mathfrak{Z}}_1}{[(0.4,0.5),(0.5,0.6)]},\frac{\breve{\mathfrak{Z}}_2}{[(0.7,0.6),(0.3,0.5)]},\frac{\breve{\mathfrak{Z}}_3}{[(0.3,0.7),(0.4,0.6)]},\frac{\breve{\mathfrak{Z}}_4}{[(0.5,0.7),(0.3,0.6)]}\right\}\\ \left.\breve{F}(\rho_3)=\left\{\frac{\breve{\mathfrak{Z}}_1}{[(0.3,0.3),(0.6,0.5)]},\frac{\breve{\mathfrak{Z}}_2}{[(0.7,0.3),(0.5,0.4)]},\frac{\breve{\mathfrak{Z}}_3}{[(0.5,0.5),(0.6,0.5)]},\frac{\breve{\mathfrak{Z}}_4}{[(0.7,0.6),(0.3,0.5)]}\right\}\right\}\\ \subseteq L_q * q - ROMFSS.\end{array}\right\}$$

Definition 2.6 [40] Let $\breve{F}(\rho_1)=\{\langle\breve{\mathfrak{z}},\ (\mu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\nu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\ (\mu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\nu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\ldots,(\mu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\rangle:\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}\}$ and $\breve{F}(\rho_2)=\{\langle\ \breve{\mathfrak{z}},\ (\mu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),\nu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),(\mu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),\nu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),\ldots,(\mu^r_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}),\nu^r_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\rangle:\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}\}\in(\breve{F},\Delta)$, for all $\rho_1,\rho_2\in\Delta$. We demonstrate the following

(i) $\breve{F}(\rho_1)=\breve{F}(\rho_2)$ iff $\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})=\mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})$ and $\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})=\nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})$, for all $m=1,2,\ldots,r$ and $\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}$.

(ii) $\breve{F}(\rho_1)\breve{\subseteq}\breve{F}(\rho_2)$ iff $\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\le\mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})$ and $\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\ge\nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})$, for all $m=1,2,\ldots,r$ and $\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}$.

(iii) $$\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)=\left\{\begin{array}{c}\Big\langle x,((\mu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\vee}\mu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})),\ldots,(\mu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\vee}\mu^r_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}))),\\ ((\nu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\vee}\nu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})),\cdots,(\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\vee}\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})))\Big\rangle:\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}\end{array}\right\}.$$

(iv) $$\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)=\left\{\begin{array}{c}\Big\langle x,\Big((\mu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\wedge}\mu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})),\ldots,(\mu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\wedge}\mu^r_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}))\Big),\\ \Big((\nu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\wedge}\nu^1_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})),\cdots,(\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\grave{\wedge}\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}))\Big)\Big\rangle:\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}\end{array}\right\}.$$

(v) $\breve{F}^c(\rho)=\left\{\langle\breve{\mathfrak{z}},(\nu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\mu^1_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})),\cdots,(\nu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}),\mu^r_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}))\rangle:\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}\right\}$

Definition 2.7 [40] The union of two L_q* q-ROMFSSs $(\breve{F},\Delta)$ and $(\breve{\mathfrak{J}},\Gamma)$ over a universe $\breve{\mathfrak{z}}$ is a L_q* q-ROMFSS $(\breve{\mathfrak{K}},\Upsilon)$, where $\Upsilon=\Delta\cup\Gamma$. For all $\rho\in\Upsilon$ and $\breve{\mathfrak{z}}\in\breve{\mathfrak{Z}}$, the membership and non-membership functions in a multi-dimensional context are defined as follows:

$$\mu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}})=\begin{cases}\mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho\in\Delta-\Gamma\\ \mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho\in\Gamma-\Delta\\ max\left\{\mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}),\mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\right\} & \text{if } \rho\in\Delta\cap\Gamma\end{cases}$$

$$\nu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = \begin{cases} \nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Delta - \Gamma \\ \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Gamma - \Delta \\ min\{\nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\} & \text{if } \rho \in \Delta \cap \Gamma \end{cases}$$

The union is denoted by $(\breve{F}, \Delta) \breve{\cup} (\breve{\mathfrak{J}}, \Gamma) = (\mathfrak{R}, \Upsilon)$

Definition 2.8 [40] The intersection of two L_q* q-ROMFSSs $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ over a universe $\breve{\mathfrak{z}}$ is a L_q* q-ROMFSS $(\breve{\mathfrak{K}}, \Upsilon)$, where $\Upsilon = \Delta \cup \Gamma$. For all $\rho \in \Upsilon$ and $\breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}$, the membership and non-membership functions in a multi-dimensional context are defined as follows:

$$\mu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = \begin{cases} \mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Delta - \Gamma \\ \mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Gamma - \Delta \\ min\{\mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\} & \text{if } \rho \in \Delta \cap \Gamma \end{cases}$$

$$\nu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = \begin{cases} \nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Delta - \Gamma \\ \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}}) & \text{if } \rho \in \Gamma - \Delta \\ max\{\nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\} & \text{if } \rho \in \Delta \cap \Gamma \end{cases}$$

The intersection is denoted by $(\breve{F}, \Delta) \breve{\cap} (\breve{\mathfrak{J}}, \Gamma) = (\breve{\mathfrak{K}}, \Upsilon)$.

Definition 2.9 [40] The restricted union of two L_q* q-ROMFSSs $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ over a universe $\breve{\mathfrak{z}}$, denoted by $(\breve{F}, \Delta) \sqcup (\breve{\mathfrak{J}}, \Gamma)$ is a L_q* q-ROMFSS $(\breve{\mathfrak{K}}, \Upsilon)$, where $\Upsilon = \Delta \cap \Gamma$. For all $\rho \in \Upsilon$ and $\breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}$.

$$\mu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = max\{\mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\} \; and \; \nu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = min\{\nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\}$$

Definition 2.10 [40] The restricted intersection of two L_q* q-ROMFSSs $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ over a universe $\breve{\mathfrak{z}}$, denoted by $(\breve{F}, \Delta) \sqcap (\breve{\mathfrak{J}}, \Gamma)$ is a L_q* q-ROMFSS $(\breve{\mathfrak{K}}, \Upsilon)$, where $\Upsilon = \Delta \cap \Gamma$. For all $\rho \in \Upsilon$ and $\breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}$.

$$\mu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = min\{\mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \mu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\} \; and \; \nu^m_{\breve{\mathfrak{K}}(\rho)}(\breve{\mathfrak{z}}) = max\{\nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}}), \nu^m_{\breve{\mathfrak{J}}(\rho)}(\breve{\mathfrak{z}})\}$$

3 Algebraic Structure of L_q* q-ROMFS Sets

Proposition 3.1 *Let $(\breve{F}, \Delta)$ be a L_q* q-ROMFSS over $\breve{\mathfrak{Z}}$ of dimension m. Then $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ and $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)$ are the least upper bound and greatest lower bound of $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$.*

Proof It is enough to show that

(i) $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ is the least upper bound of $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$.
(ii) $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)$ is the greatest lower bound of $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$.

Let $\breve{F}(\rho_1) = \{\langle \breve{\mathfrak{z}}, \mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}})\rangle : \breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}\}$ and $\breve{F}(\rho_2) = \{\langle \breve{\mathfrak{z}}, \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\rangle : \breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}\}$.
By definition 2.6.
$\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) = \{\langle \breve{\mathfrak{z}}, max\{\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}, min\{\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}\rangle : \breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}\}$.
We need to show that
$\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ is the smallest q-ROMFSS containing both $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$.

For any $\breve{\mathfrak{z}} \in \breve{\mathfrak{Z}}$, we have
$\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \leq \max\{\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}$ and $\mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}) \leq \max\{\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}$
$\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \geq \min\{\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}$ and $\nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}) \geq \min\{\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\}$.
Thus $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ and $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$.
Moreover, if $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho)$ and $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho)$, for some $\breve{F}(\rho)$.
then, $\max\{\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\} \leq \mu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}})$ and $\min\{\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}), \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}})\} \geq \nu^m_{\breve{F}(\rho)}(\breve{\mathfrak{z}})$.
Therefore, $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho)$.
Hence $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ is the least upper bound of $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$.
Similarly, we can prove that $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)$ is the greatest lower bound of $\breve{F}(\rho_1)$ and $\breve{F}(\rho_2)$. □

Proposition 3.2 *Let $(\breve{F}, \Delta)$ be a L_q* q-ROMFSS over $\breve{\mathfrak{Z}}$ of dimension m and $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Then it satisfies the following:*

(i) *Idempotent laws: $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2) = \breve{F}(\rho_1)$ and $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) = \breve{F}(\rho_1)$.*
(ii) *Absorption laws:*
$\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_1)$ *and* $\breve{F}(\rho_1) \grave{\wedge} (\breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)) = \breve{F}(\rho_1)$
(iii) *Associative laws:*
$\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)) = (\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)) \grave{\vee} \breve{F}(\rho_3)$ *and*
$\breve{F}(\rho_1) \grave{\wedge} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = (\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)) \grave{\wedge} \breve{F}(\rho_3)$
(iv) *Commutative laws:*
$\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2) = \breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_1)$ *and* $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) = \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_1)$

Proof The proof is obvious from the Definition 2.4. □

Definition 3.3 Let $(\breve{F}, \Delta)$ be a L_q* q-ROMFS set of dimension m over $\breve{\mathfrak{Z}}$. Then $(\breve{F}, \Delta)$ is called L_q* q-ROMFS chain if for all $\breve{F}(\rho_1), \breve{F}(\rho_2) \in (\breve{F}, \Delta)$, we have either $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ or $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_1)$, for all $\rho_1, \rho_2 \in \Delta$.

Example 3.4 Consider the universal set $\breve{\mathfrak{z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3\}$ and the parameter set $\Delta = \{\rho_1, \rho_2, \rho_3\}$ with q = 2. Define the q-ROMFS set $(\breve{F}, \Delta)$ of dimension m =1 as follows:

$$(\breve{F}, \Delta) = \left\{ \begin{array}{l} \breve{F}(\rho_1) = \{\langle \breve{\mathfrak{z}}_1, (0.2, 0.7)\rangle, \langle \breve{\mathfrak{z}}_2, (0.3, 0.6)\rangle, \langle \breve{\mathfrak{z}}_3, (0.4, 0.5)\rangle\} \\ \breve{F}(\rho_2) = \{\langle \breve{\mathfrak{z}}_1, (0.3, 0.6)\rangle, \langle \breve{\mathfrak{z}}_2, (0.4, 0.5)\rangle, \langle \breve{\mathfrak{z}}_3, (0.5, 0.4)\rangle\} \\ \breve{F}(\rho_3) = \{\langle \breve{\mathfrak{z}}_1, (0.4, 0.5)\rangle, \langle \breve{\mathfrak{z}}_2, (0.5, 0.4)\rangle, \langle \breve{\mathfrak{z}}_3, (0.6, 0.3)\rangle\} \end{array} \right\}$$

For any $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3)$ in $(\breve{F}, \Delta)$ implies $\mu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \leq \mu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}) \leq \mu^m_{\breve{F}(\rho_3)}(\breve{\mathfrak{z}})$, $\nu^m_{\breve{F}(\rho_1)}(\breve{\mathfrak{z}}) \geq \nu^m_{\breve{F}(\rho_2)}(\breve{\mathfrak{z}}) \geq \nu^m_{\breve{F}(\rho_3)}(\breve{\mathfrak{z}})$, for all $\breve{\mathfrak{z}} \in \breve{\mathfrak{z}}$.
Therefore $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_3)$.
Hence $(\breve{F}, \Delta)$ forms a q-ROMFS chain (Fig. 2).

Proposition 3.5 *Every q-ROMFS chain is an L_q* q-ROMFSS over $\breve{\mathfrak{Z}}$.*

Proof Let $(\breve{F}, \Delta)$ be a q-ROMFS chain. Since any two elements are comparable. i.e., $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ or $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_1)$, for all $\rho_1, \rho_2 \in \Delta$. It follows that $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ and $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)$ exists in $(\breve{F}, \Delta)$. Hence $(\breve{F}, \Delta)$ is a L_q* q-ROMFSS over $\breve{\mathfrak{Z}}$. □

Definition 3.6 A modular L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$ $(\breve{F}, \Delta)$ is defined as follows: If $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ implies $\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_2) \grave{\wedge} (\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3))$ for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$, then it is termed as a modular L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$.

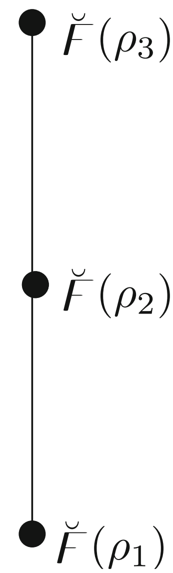

Fig. 2 L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) chain

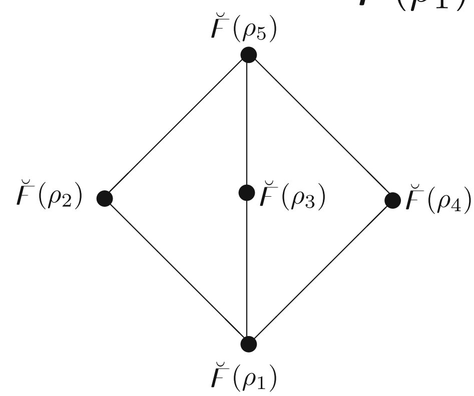

Fig. 3 Modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$)

Example 3.7 Let $\breve{\mathfrak{z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3\}$ denote the universal set, and $\Delta = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\} \grave{\subseteq} \mathfrak{B}$ represent a lattice of parameters. We define a L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) as follows:

$$(\breve{F}, \Delta) = \left\{ \begin{array}{r} \breve{F}(\rho_1) = \big\{\langle\breve{\mathfrak{z}}_1, (0.55, 0.51), (0.49, 0.57), (0.42, 0.72)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.26, 0.65), (0.20, 0.74), (0.17, 0.88)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.43, 0.65), (0.38, 0.69), (0.21, 0.88)\big\rangle\big\} \\ \breve{F}(\rho_2) = \big\{\langle\breve{\mathfrak{z}}_1, (0.59, 0.48), (0.53, 0.53), (0.46, 0.65)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.39, 0.42), (0.33, 0.52), (0.25, 0.73)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.64, 0.59), (0.57, 0.61), (0.49, 0.72)\big\rangle\big\} \\ \breve{F}(\rho_3) = \big\{\langle\breve{\mathfrak{z}}_1, (0.63, 0.45), (0.54, 0.53), (0.47, 0.64)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.54, 0.31), (0.47, 0.49), (0.42, 0.64)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.79, 0.48), (0.72, 0.53), (0.61, 0.64)\big\rangle\big\} \\ \breve{F}(\rho_4) = \big\{\langle\breve{\mathfrak{z}}_1, (0.72, 0.42), (0.68, 0.51), (0.54, 0.63)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.65, 0.20), (0.58, 0.32), (0.49, 0.52)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.81, 0.35), (0.74, 0.41), (0.65, 0.59)\big\rangle\big\} \\ \breve{F}(\rho_5) = \big\{\langle\breve{\mathfrak{z}}_1, (0.88, 0.21), (0.81, 0.32), (0.72, 0.43)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.72, 0.15), (0.68, 0.24), (0.51, 0.47)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.93, 0.21), (0.87, 0.32), (0.69, 0.53)\big\rangle\big\} \end{array} \right\}$$

Therefore, $(\breve{F}, \Delta)$ exhibits a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Figure 3 illustrates the lattice structure of the parameters considered.

Proposition 3.8 *In a L_q* q-ROMFSS($\breve{\mathfrak{Z}}$), the following expressions are equivalent:*

(i) $(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2))\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_1)\grave{\wedge}((\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2))\grave{\vee}\breve{F}(\rho_3))$

(ii) $(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2))\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)))$

(iii) $(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2))\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_1)\grave{\vee}((\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)))\grave{\wedge}\breve{F}(\rho_3))$

(iv) $(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2))\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)))$

(v) $\breve{F}(\rho_1)\grave{\subseteq}\breve{F}(\rho_2)$ *implies that* $\breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)))$

(vi) $\breve{F}(\rho_1) \supseteq \breve{F}(\rho_2)$ *implies that* $\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)))$

If any of the aforementioned conditions is satisfied within a L_q* q-ROMFSS($\breve{\mathfrak{Z}}$), then the L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is classified as a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$).

Proof (I) To prove: (i) and (ii) are equivalent.
Consider $(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2))\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3))$
Using distributivity of $\grave{\wedge}$ over $\grave{\vee}$:
$\breve{F}(\rho_1)\grave{\wedge}((\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2))\grave{\vee}\breve{F}(\rho_3))$
Rewriting the terms, the equivalence follows.
(II) To prove: (iii) and (iv) are equivalent.
Consider $(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2))\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$
Using distributivity of $\grave{\vee}$ over $\grave{\wedge}$:
$\breve{F}(\rho_1)\grave{\vee}((\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3))\grave{\vee}\breve{F}(\rho_1))$
Rewriting and grouping the terms, the equivalence follows.
(III) For (v), If $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$, then: $\breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2))\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}((\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$
For (vi), If $\breve{F}(\rho_1) \grave{\supseteq} \breve{F}(\rho_2)$, using lattice symmetry: $\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2))\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\vee}((\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3))$
Thus the equivalence follows.
If any condition (i)-(vi) holds, then it satisfies the modular law, classifying the L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) as modular. □

Definition 3.9 A dual form of modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is written as if $\breve{F}(\rho_1) \grave{\supseteq} \breve{F}(\rho_2)$ implies $\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3))$, for all $\breve{F}(\rho_2), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$.

Proposition 3.10 *The sublattice(dual) of a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is modular.*

Proof Let $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Then $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2) \implies \breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Let $(\breve{F}_0, \Delta_0)$ be a sublattice of $(\breve{F}, \Delta)$. Take $\breve{F}(\rho_1)$, $\breve{F}(\rho_2)$, $\breve{F}(\rho_3) \in (\breve{F}_0, \Delta_0)$. Therefore, $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2) \implies \breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$ is true for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Hence the sublattice is a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Therefore the sublattice of a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is modular.
Let $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Then $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2) \implies \breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. The dual of $(\breve{F}, \Delta)$ is $\breve{F}(\rho_1) \grave{\supseteq} \breve{F}(\rho_2) \implies \breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3))$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Therefore the dual of a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is modular. □

Proposition 3.11 *In a L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) $(\breve{F}, \Delta)$, the following conditions are equivalent:*

(i) $(\breve{F}, \Delta)$ *is a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$),*

(ii) $\breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)))\grave{\wedge}\breve{F}(\rho_3)$

(iii) *L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) does not contain a pentagon.*

Proposition 3.12 *If* $(\breve{F}, \Delta)$ *be a modular* L_q* *q-ROMFSS*$(\breve{\mathfrak{Z}})$*, then* $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$*,* $\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_3) = \breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3))$ *and* $\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3) = \breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3))$ *imply* $\breve{F}(\rho_1) = \breve{F}(\rho_2)$*, for all* $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$*.*

Proof Let $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$. Then for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$, if $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$, it follows that $\breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)) = \breve{F}(\rho_2)\grave{\wedge}(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3))$. Now

$$\begin{aligned}\breve{F}(\rho_1) &= \breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\ &= \breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\\ &= \breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\ &= \breve{F}(\rho_2)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\ &= \breve{F}(\rho_2)\end{aligned}$$

□

Theorem 3.13 *A* L_q* *q-ROMFSS*$(\breve{\mathfrak{Z}})$ $(\breve{F}, \Delta)$ *is modular iff for all* $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$ *with* $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$*,* $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big) = \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)$*.*

Proof Suppose that $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$.
To prove that for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$ with $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$, $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big) = \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)$. Now

$$\begin{aligned}&\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\ &\quad= \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\ &\quad= \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\ &\quad= \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big)\end{aligned} \tag{1}$$

$$\begin{aligned}&\breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\ &\quad= \breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\ &\quad= \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\ &\quad= \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)\end{aligned} \tag{2}$$

From (1) and (2), we get $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big) = \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)$ $\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)$.
Conversely, if $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$, then

$$\begin{aligned}&\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big)\\ &\quad= \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\grave{\vee}\big(\breve{F}(\rho_3)\grave{\wedge}\breve{F}(\rho_1)\big)\\ &\quad= \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\end{aligned} \tag{3}$$

$$\begin{aligned}&\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)\\ &\quad= \breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_1)\big)\\ &\quad= \breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\end{aligned} \tag{4}$$

From (1) and (2), we get $\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)$.
Hence $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$. □

Definition 3.14 A distributive L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$ $(\breve{F}, \Delta)$ is defined as follows: If $\breve{F}(\rho_1) \grave{\vee} \big(\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)\big) = \breve{F}(\rho_2) \grave{\wedge} \big(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3)\big)$ for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$, then it is termed as a distributive L_q* q-ROMFSS$(\breve{\mathfrak{Z}})$.

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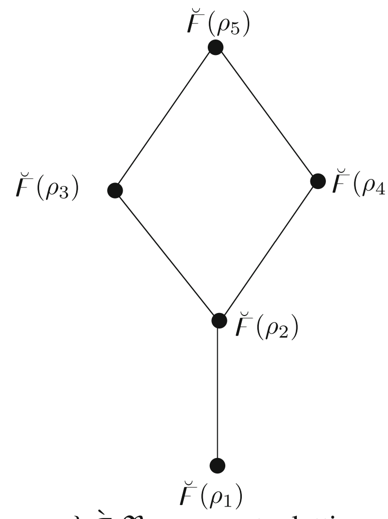
Fig. 4 Distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$)



Example 3.15 Let $\breve{\mathfrak{z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3\}$ denote the universal set, and $\Delta = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\} \grave{\subseteq} \mathfrak{B}$ represent a lattice of parameters. We define a L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) as follows:

$$(\breve{F}, \Delta) = \left\{\begin{array}{r} \breve{F}(\rho_1) = \big\{\langle\breve{\mathfrak{z}}_1, (0.55, 0.51), (0.49, 0.57), (0.42, 0.72)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle\big\} \\ \breve{F}(\rho_2) = \big\{\langle\breve{\mathfrak{z}}_1, (0.59, 0.48), (0.53, 0.53), (0.46, 0.65)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle\big\} \\ \breve{F}(\rho_3) = \big\{\langle\breve{\mathfrak{z}}_1, (0.63, 0.45), (0.54, 0.53), (0.47, 0.64)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle\big\} \\ \breve{F}(\rho_4) = \big\{\langle\breve{\mathfrak{z}}_1, (0.72, 0.42), (0.68, 0.51), (0.54, 0.63)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle\big\} \\ \breve{F}(\rho_5) = \big\{\langle\breve{\mathfrak{z}}_1, (0.81, 0.32), (0.88, 0.21), (0.72, 0.43)\big\rangle, \\ \big\langle\breve{\mathfrak{z}}_2, (0.72, 0.15), (0.68, 0.24), (0.51, 0.47)\big\rangle \\ , \big\langle\breve{\mathfrak{z}}_3, (0.93, 0.21), (0.87, 0.32), (0.69, 0.53)\big\rangle\big\} \end{array}\right\}$$

Therefore, $(\breve{F}, \Delta)$ exhibits a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Figure 4 illustrates the lattice structure of the parameters considered.

Definition 3.16 A dual form of distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is written as if for all $\breve{F}(\rho_2), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$, $\breve{F}(\rho_1)\dot{\wedge}(\breve{F}(\rho_2)\dot{\vee}\breve{F}(\rho_3)) = (\breve{F}(\rho_1)\dot{\wedge}\breve{F}(\rho_2))\dot{\vee}(\breve{F}(\rho_3)\dot{\wedge}\breve{F}(\rho_3))$.

Proposition 3.17 *Dual of a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is distributive.*

Proposition 3.18 *The sublattice of a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is distributive.*

Proposition 3.19 *A L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) $(\breve{F}, \Delta)$ is distributive iff L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) does not contain a pentagon or a diamond.*

Proposition 3.20 *A L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) $(\breve{F}, \Delta)$ is distributive iff $\breve{F}(\rho_1)\dot{\wedge}(\breve{F}(\rho_2)\dot{\vee}\breve{F}(\rho_3)) = (\breve{F}(\rho_1)\dot{\wedge}\breve{F}(\rho_2))\dot{\vee}\big(\breve{F}(\rho_3)\dot{\wedge}\breve{F}(\rho_3)\big)$ for all $\breve{F}(\rho_2), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$.*

Proof Let $(\breve{F}, \Delta)$ is distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Then $\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = (\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2))\grave{\wedge}\big(\breve{F}(\rho_3)\grave{\vee}\breve{F}(\rho_3)\big)$ for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Now

$$\begin{aligned}
&\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\wedge}\big(\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_1)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_1)\big)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_1)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3))
\end{aligned} \tag{5}$$

Conversely,

$$\begin{aligned}
&\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\vee}\big(\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_1)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_1)\big)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\big)\\
&\quad = \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_1)\big)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&\quad = \breve{F}(\rho_1)\grave{\vee}(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3))
\end{aligned} \tag{6}$$

□

Proposition 3.21 *Every q-ROMFS chain is a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$).*

Proof Let $(\breve{F}, \Delta)$ be a multi-fuzzy soft chain. Here for all $\rho_1, \rho_2 \in \Delta$, $\rho_1 \geq \rho_2$ implies that $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$. We will prove the results in three cases:
Case I. $\rho_1 \geq \rho_2$ and $\rho_1 \grave{\subseteq} \rho_2 \grave{\vee} \rho_3$ implies $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ and $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Then $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$. Therefore

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_1) \tag{7}$$

As $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ implies $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2) = \breve{F}(\rho_1)$. Then $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_1)$. Therefore

$$\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \tag{8}$$

From (7) and (8), we get,

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)$$

Case II. $\rho_1 \geq \rho_2$ and $\rho_1 \grave{\subseteq} \rho_2 \grave{\vee} \rho_3$ implies $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2)$ and $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Then $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$. Therefore

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_1) \tag{9}$$

As $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2)$ implies $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2) = \breve{F}(\rho_1)$. Then $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_1)$. Therefore

$$\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \tag{10}$$

From (9) and (10), we get,

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)$$

Case III. $\rho_1 \geq \rho_2$ and $\rho_1 \dot{\subseteq} \rho_2 \grave{\vee} \rho_3$ implies $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2)$ and $\breve{F}(\rho_2) \dot{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Then $\breve{F}(\rho_2) \dot{\subseteq} \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$. Therefore

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \breve{F}(\rho_1) \tag{11}$$

As $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2)$ implies $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2) = \breve{F}(\rho_1)$.
Then $\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_1)$. Therefore

$$\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big) = \breve{F}(\rho_1) \tag{12}$$

From (11) and (12), we get,

$$\breve{F}(\rho_1)\grave{\wedge}(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)) = \big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_2)\big)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)$$

□

Proposition 3.22 *Every distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) is a modular L_q* q-ROMFSS($\breve{\mathfrak{Z}}$).*

Proof Let $(\breve{F}, \Delta)$ be a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Then $\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = \big(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)\big) \grave{\wedge} \big(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3)\big)$ for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. If $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2)$, then $\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = \breve{F}(\rho_2) \grave{\wedge} \big(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3)\big)$ (since $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) = \breve{F}(\rho_2)$). Hence establishing the modularity. □

Remark 3.23 Converse of the above Proposition 3.22 need not be true.

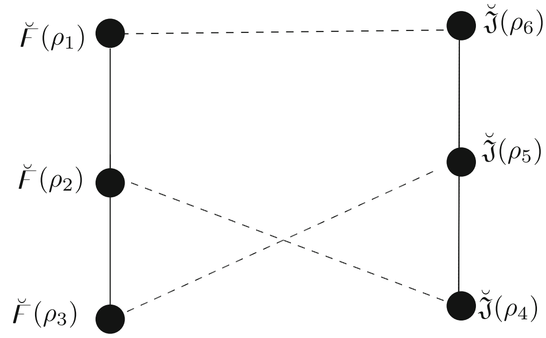
Proposition 3.24 *If L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) $(\breve{F}, \Delta)$ is a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$), then $\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_3) = \breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)$ and $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3) = \breve{F}(\rho_2) \grave{\vee} \breve{F}(\rho_3)$ imply $\breve{F}(\rho_1) = \breve{F}(\rho_2)$, for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$.*

Proof Let $(\breve{F}, \Delta)$ be a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$). Then $\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3)) = \big(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)\big) \grave{\wedge} (\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_3))$ for all $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$. Now

$$\begin{aligned}
\breve{F}(\rho_1) &= \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&= \breve{F}(\rho_1)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&= \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_3)\big)\\
&= \big(\breve{F}(\rho_1)\grave{\vee}\breve{F}(\rho_2)\big)\grave{\wedge}\big(\breve{F}(\rho_2)\grave{\vee}\breve{F}(\rho_3)\big)\\
&= \breve{F}(\rho_2)\grave{\vee}\big(\breve{F}(\rho_1)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&= \breve{F}(\rho_2)\grave{\vee}\big(\breve{F}(\rho_2)\grave{\wedge}\breve{F}(\rho_3)\big)\\
&= \breve{F}(\rho_2)
\end{aligned}$$

□

Fig. 5 Homomorphism of two L_q* q-ROMFS sets



Proposition 3.25 *A L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) $(\breve{F}, \Delta)$ is a distributive L_q* q-ROMFSS($\breve{\mathfrak{Z}}$) iff $\big(\breve{F}(\rho_1) \dot{\vee} \breve{F}(\rho_2)\big) \dot{\wedge} \big(\breve{F}(\rho_2) \dot{\vee} \breve{F}(\rho_3)\big) \dot{\wedge} \big(\breve{F}(\rho_3) \dot{\vee} \breve{F}(\rho_1)\big) = \big(\breve{F}(\rho_1) \dot{\wedge} \breve{F}(\rho_2)\big) \dot{\vee} \big(\breve{F}(\rho_2) \dot{\wedge} \breve{F}(\rho_3)\big) \dot{\vee} \big(\breve{F}(\rho_3) \dot{\wedge} \breve{F}(\rho_1)\big)$, for all $\breve{F}(\rho_1)$, $\breve{F}(\rho_2)$, $\breve{F}(\rho_3) \in (\breve{F}, \Delta)$.*

3.1 Homomorphism of L_q* q-ROMFS Set

Definition 3.26 Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be L_q* q-ROMFS sets. A mapping $\breve{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is defined as meet homomorphism. If $\breve{\mathfrak{f}}\,\big(\breve{F}(\rho_1) \dot{\vee} \breve{F}(\rho_2)\big) = \breve{\mathfrak{f}}\,\big(\breve{F}(\rho_1)\big) \dot{\vee} \breve{\mathfrak{f}}\,\big(\breve{F}(\rho_2)\big)$.
It is termed a join homomorphism. If $\breve{\mathfrak{f}}\,\big(\breve{F}(\rho_1) \dot{\wedge} \breve{F}(\rho_2)\big) = \breve{\mathfrak{f}}\,\big(\breve{F}(\rho_1)\big) \dot{\wedge} \breve{\mathfrak{f}}\,\big(\breve{F}(\rho_2)\big)$.
If $\breve{\mathfrak{f}}$ functions as both a meet and join homomorphism, it is referred to as an L_q* q-ROMFS homomorphism.

Example 3.27 Let $\breve{\mathfrak{z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3\}$ be the universal set and $\mathfrak{E} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$ be a lattice of attributes and $\Delta, \Gamma \subseteq \mathfrak{E}$, $\Delta = \{\rho_1, \rho_2, \rho_3\}$; $\Gamma = \{\rho_4, \rho_5, \rho_6\}$. We consider the two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$) defined as follows:

$$(\breve{F}, \Delta) = \left\{ \begin{array}{r} \breve{F}(\rho_1) = \big\{\langle \breve{\mathfrak{z}}_1, (0.5, 0.6), (0.4, 0.7)\rangle, \langle \breve{\mathfrak{z}}_2, (0.6, 0.5), (0.5, 0.8)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.7, 0.4), (0.6, 0.6)\rangle\big\} \\ \breve{F}(\rho_2) = \big\{\langle \breve{\mathfrak{z}}_1, (0.7, 0.2), (0.6, 0.4)\rangle, \langle \breve{\mathfrak{z}}_2, (0.8, 0.3), (0.7, 0.5)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.8, 0.3), (0.7, 0.1)\rangle\big\} \\ \breve{F}(\rho_3) = \big\{\langle \breve{\mathfrak{z}}_1, (0.8, 0.1), (0.7, 0.3)\rangle, \langle \breve{\mathfrak{z}}_2, (0.9, 0.2), (0.8, 0.4)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.85, 0.1), (0.7, 0.2)\rangle\big\} \end{array} \right\}$$

and

$$(\breve{\mathfrak{J}}, \Gamma) = \left\{ \begin{array}{r} \breve{\mathfrak{J}}(\rho_4) = \big\{\langle \breve{\mathfrak{z}}_1, (0.5, 0.4), (0.4, 0.5)\rangle, \langle \breve{\mathfrak{z}}_2, (0.4, 0.5), (0.3, 0.8)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.5, 0.4), (0.4, 0.7)\rangle\big\} \\ \breve{\mathfrak{J}}(\rho_5) = \big\{\langle \breve{\mathfrak{z}}_1, (0.7, 0.2), (0.6, 0.4)\rangle, \langle \breve{\mathfrak{z}}_2, (0.6, 0.3), (0.4, 0.6)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.7, 0.3), (0.6, 0.5)\rangle\big\} \\ \breve{\mathfrak{J}}(\rho_6) = \big\{\langle \breve{\mathfrak{z}}_1, (0.9, 0.2), (0.7, 0.3)\rangle, \langle \breve{\mathfrak{z}}_2, (0.8, 0.1), (0.6, 0.2)\rangle, \\ \langle \breve{\mathfrak{z}}_3, (0.85, 0.15), (0.7, 0.4)\rangle\big\} \end{array} \right\}$$

Let $\breve{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ be the map given by $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) = \breve{\mathfrak{J}}(\rho_4)$, $\breve{\mathfrak{f}}(\breve{F}(\rho_2)) = \breve{\mathfrak{J}}(\rho_6)$ and $\breve{\mathfrak{f}}(\breve{F}(\rho_3)) = \breve{\mathfrak{J}}(\rho_5)$. Thus the map $\breve{\mathfrak{f}}$ of Fig. 5 is a q-ROMFS homomorphism.

Definition 3.28 Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). Then a map $\breve{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is said to be a L_q* q-ROMFS isotone map if $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2) \Longrightarrow \breve{\mathfrak{f}}(\breve{F}(\rho_1)) \dot{\subseteq} \breve{\mathfrak{f}}(\breve{F}(\rho_2))$.

Proposition 3.29 *Every L_q* (Lattice Ordered) q- rung orthopair multi-fuzzy soft homomorphism is a L_q* q-ROMFS isotone map.*

Proof Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). Let $\breve{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ be a L_q* q-ROMFS homomorphism from a L_q* q-ROMFSS($\breve{\mathfrak{z}}$ $(\breve{F}, \Delta)$ to $(\breve{\mathfrak{J}}, \Gamma)$. Let $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_2) \in (\breve{F}, \Delta)$, then $\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2) = \breve{F}(\rho_2)$. Then $\breve{\mathfrak{f}}(\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)) = \breve{\mathfrak{f}}(\breve{F}(\rho_2))$. Hence $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\vee} \breve{\mathfrak{f}}(\breve{F}(\rho_2)) = \breve{\mathfrak{f}}(\breve{F}(\rho_2))$, so $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\subseteq} \breve{\mathfrak{f}}(\breve{F}(\rho_2))$ is $(\breve{\mathfrak{J}}, \Gamma)$. Thus $\breve{\mathfrak{f}}$ is a L_q* q-ROMFS isotone map. □

Remark 3.30 Converse of the above Proposition 3.29 need not be true. Let $\breve{\mathfrak{z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3\}$ be the universal set and $\mathfrak{B} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$ be a lattice of parameters and $\Delta, \Gamma \grave{\subseteq} \mathfrak{B}$, $\Delta = \{\rho_1, \rho_2, \rho_3\}$, $\Gamma = \{\rho_4, \rho_5, \rho_6\}$. We consider the two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$) defined as follows:

$$(\breve{F}, \Delta) = \left\{\begin{array}{r} \breve{F}(\rho_1) = \{\langle \breve{\mathfrak{z}}_1, (0.32, 0.81), (0.42, 0.74), (0.56, 0.63)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.28, 0.93), (0.34, 0.85), (0.54, 0.64)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.15, 0.97), (0.31, 0.82), (0.44, 0.71)\rangle\} \\ \breve{F}(\rho_2) = \{\langle \breve{\mathfrak{z}}_1, (0.43, 0.79), (0.55, 0.68), (0.65, 0.42)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.35, 0.81), (0.43, 0.76), (0.56, 0.57)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.23, 0.92), (0.39, 0.77), (0.57, 0.52)\rangle\} \\ \breve{F}(\rho_3) = \{\langle \breve{\mathfrak{z}}_1, (0.59, 0.61), (0.66, 0.53), (0.72, 0.36)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.59, 0.63), (0.64, 0.52), (0.73, 0.41)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.61, 0.81), (0.73, 0.65), (0.86, 0.42)\rangle\} \end{array}\right\}$$

and

$$(\breve{\mathfrak{J}}, \Gamma) = \left\{\begin{array}{r} \breve{\mathfrak{J}}(\rho_4) = \{\langle \breve{\mathfrak{z}}_1, (0.25, 0.92), (0.42, 0.86), (0.58, 0.73)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.18, 0.73), (0.25, 0.65), (0.32, 0.59)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.12, 0.85), (0.34, 0.72), (0.44, 0.62)\rangle\} \\ \breve{\mathfrak{J}}(\rho_5) = \{\langle \breve{\mathfrak{z}}_1, (0.39, 0.85), (0.54, 0.71), (0.67, 0.62)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.23, 0.64), (0.36, 0.58), (0.48, 0.42)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.21, 0.64), (0.32, 0.57), (0.58, 0.50)\rangle\} \\ \breve{\mathfrak{J}}(\rho_6) = \{\langle \breve{\mathfrak{z}}_1, (0.45, 0.72), (0.63, 0.59), (0.81, 0.42)\rangle, \\ \langle \breve{\mathfrak{z}}_2, (0.38, 0.61), (0.48, 0.52), (0.62, 0.35)\rangle \\ , \langle \breve{\mathfrak{z}}_3, (0.34, 0.52), (0.49, 0.43), (0.69, 0.32)\rangle\} \end{array}\right\}$$

Proposition 3.31 *Homomorphic image of a modular L_q* q-ROMFSS($\breve{\mathfrak{z}}$) is modular.*

Proof Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). Let $(\breve{F}, \Delta)$ be a modular L_q* q-ROMFSS($\breve{\mathfrak{z}}$) and $(\breve{\mathfrak{J}}, \Gamma)$ be a lattice ordered q-ROMFS homomorphic image of $(\breve{F}, \Delta)$. Let $\breve{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ be an onto lattice ordered q-ROMFS homomorphism. Let $\breve{\mathfrak{J}}(\varpi_1), \breve{\mathfrak{J}}(\varpi_2), \breve{\mathfrak{J}}(\varpi_3) \in (\breve{\mathfrak{J}}, \Gamma)$. Since $\breve{\mathfrak{f}}$ is onto, there exist $\breve{F}(\rho_1), \breve{F}(\rho_2), \breve{F}(\rho_3) \in (\breve{F}, \Delta)$ such that $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) = \breve{\mathfrak{J}}(\varpi_1)$, $\breve{\mathfrak{f}}(\breve{F}(\rho_2)) = \breve{\mathfrak{J}}(\varpi_2)$, $\breve{\mathfrak{f}}(\breve{F}(\rho_3)) = \breve{\mathfrak{J}}(\varpi_3)$. Let $\breve{\mathfrak{J}}(\varpi_1) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_3)$,

$$\begin{aligned} \breve{\mathfrak{J}}(\varpi_1) \grave{\vee} (\breve{\mathfrak{J}}(\varpi_2) \grave{\wedge} \breve{\mathfrak{J}}(\varpi_3)) &= \breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\vee} (\breve{\mathfrak{f}}(\breve{F}(\rho_2)) \grave{\wedge} \breve{\mathfrak{f}}(\breve{F}(\rho_3))) \\ &= \breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\vee} (\breve{\mathfrak{f}}(\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3))) \\ &= \breve{\mathfrak{f}}(\breve{F}(\rho_1) \grave{\vee} (\breve{F}(\rho_2) \grave{\wedge} \breve{F}(\rho_3))) \\ &= \breve{\mathfrak{f}}((\breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)) \grave{\wedge} \breve{F}(\rho_3)) \end{aligned}$$

$$= \check{\mathfrak{f}}((\breve{F}(\rho_1)\dot{\vee}\breve{F}(\rho_2))\dot{\wedge}\check{\mathfrak{f}}(\breve{F}(\rho_3))$$
$$= (\check{\mathfrak{f}}(\breve{F}(\rho_1)\dot{\vee}\breve{F}(\rho_2)))\dot{\wedge}\check{\mathfrak{f}}(\breve{F}(\rho_3))$$
$$= (\breve{\mathfrak{J}}(\varpi_1)\dot{\vee}(\breve{\mathfrak{J}}(\varpi_2))\dot{\wedge}\breve{\mathfrak{J}}(\varpi_3)$$

Thus $\breve{\mathfrak{J}}(\varpi_1) \dot{\subseteq} \breve{\mathfrak{J}}(\varpi_3) \Longrightarrow \breve{\mathfrak{J}}(\varpi_1)\dot{\vee}(\breve{\mathfrak{J}}(\varpi_2)\dot{\wedge}\breve{\mathfrak{J}}(\varpi_3)) = (\breve{\mathfrak{J}}(\varpi_1)\dot{\vee}(\breve{\mathfrak{J}}(\varpi_2))\dot{\wedge}\breve{\mathfrak{J}}(\varpi_3)$. Hence $(\breve{\mathfrak{J}}, \Gamma)$ is a modular L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). □

Proposition 3.32 *Homomorphic image of a distributive L_q* q-ROMFSS($\breve{\mathfrak{z}}$) is distributive.*

Proof Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). Let $(\breve{F}, \Delta)$ be a distributive L_q* q-ROMFSS($\breve{\mathfrak{z}}$) and $(\breve{\mathfrak{J}}, \Gamma)$ be a lattice ordered q-ROMFS homomorphic image of $(\breve{F}, \Delta)$. Let $\check{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ be an onto lattice ordered q-ROMFS homomorphism. Let $\breve{\mathfrak{J}}(\varpi_1)$, $\breve{\mathfrak{J}}(\varpi_2)$, $\breve{\mathfrak{J}}(\varpi_3) \in (\breve{\mathfrak{J}}, \Gamma)$. Since $\check{\mathfrak{f}}$ is onto, there exist $\breve{F}(\rho_1)$, $\breve{F}(\rho_2)$, $\breve{F}(\rho_3) \in (\breve{F}, \Delta)$ such that $\check{\mathfrak{f}}(\breve{F}(\rho_1)) = \breve{\mathfrak{J}}(\varpi_1)$, $\check{\mathfrak{f}}(\breve{F}(\rho_2)) = \breve{\mathfrak{J}}(\varpi_2)$, $\check{\mathfrak{f}}(\breve{F}(\rho_3)) = \breve{\mathfrak{J}}(\varpi_3)$.

$$\begin{aligned}
\breve{\mathfrak{J}}(\varpi_1)\dot{\vee}(\breve{\mathfrak{J}}(\varpi_2)\dot{\wedge}\breve{\mathfrak{J}}(\varpi_3)) &= \check{\mathfrak{f}}(\breve{F}(\rho_1))\dot{\vee}(\check{\mathfrak{f}}(\breve{F}(\rho_2))\dot{\wedge}\check{\mathfrak{f}}(\breve{F}(\rho_3))) \\
&= \check{\mathfrak{f}}(\breve{F}(\rho_1))\dot{\vee}(\check{\mathfrak{f}}(\breve{F}(\rho_2)\dot{\wedge}\breve{F}(\rho_3))) \\
&= \check{\mathfrak{f}}(\breve{F}(\rho_1)\dot{\vee}(\breve{F}(\rho_2)\dot{\wedge}\breve{F}(\rho_3))) \\
&= \check{\mathfrak{f}}((\breve{F}(\rho_1)\dot{\vee}\breve{F}(\rho_2))\dot{\wedge}(\breve{F}(\rho_1)\dot{\vee}\breve{F}(\rho_3)) \\
&= \check{\mathfrak{f}}((\breve{F}(\rho_1)\dot{\vee}\mathfrak{R}(\rho_2))\dot{\wedge}\check{\mathfrak{f}}(\breve{F}(\rho_1)\dot{\vee}\breve{F}(\rho_3)) \\
&= (\check{\mathfrak{f}}(\breve{F}(\rho_1))\dot{\vee}\check{\mathfrak{f}}(\breve{F}(\rho_2)))\dot{\wedge}(\check{\mathfrak{f}}(\breve{F}(\rho_1))\dot{\vee}\check{\mathfrak{f}}(\breve{F}(\rho_3))) \\
&= (\breve{\mathfrak{J}}(\varpi_1)\dot{\vee}(\breve{\mathfrak{J}}(\varpi_2))\dot{\wedge}(\breve{\mathfrak{J}}(\varpi_1)\dot{\vee}\breve{\mathfrak{J}}(\varpi_3))
\end{aligned}$$

Hence $(\breve{\mathfrak{J}}, \Gamma)$ is a distributive L_q* q-ROMFSS($\breve{\mathfrak{z}}$). □

Definition 3.33 Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$). Then a lattice ordered q-ROMFS homomorphism $\check{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is said to be a lattice ordered q-rung orthoapair multi-fuzzy soft isomorphism if $\check{\mathfrak{f}}$ is bijective.

If there exists a lattice ordered q-ROMFS isomorphism between two L_q* q-ROMFSSs($\breve{\mathfrak{z}}$) $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$, then the L_q* q-ROMFSSs($\breve{\mathfrak{z}}$) are said to be lattice ordered q-ROMFS isomorphic. i.e., $(\breve{F}, \Delta) \cong (\breve{\mathfrak{J}}, \Gamma)$.

Example 3.34 Consider the example 3.27, in which $\check{\mathfrak{f}}$ is bijective. Hence it is an example of lattice ordered q-ROMFS isomorphism.

Definition 3.35 Let $(\breve{F}, \Delta)$ and $(\breve{\mathfrak{J}}, \Gamma)$ be two L_q* q-ROMFSS($\breve{\mathfrak{z}}$) and $\check{\mathfrak{f}} : (\breve{F}, \Delta) \rightarrow (\breve{\mathfrak{J}}, \Gamma)$ be a Lq^* q-ROMFS homomorphism. Let K = $\{\breve{F}(\rho_1)/\breve{F}(\rho_1) \in (\breve{F}, \Delta), \check{\mathfrak{f}}(\breve{F}(\rho_1)) = \breve{\mathfrak{J}}(\varpi_\theta)\}$, where $\breve{\mathfrak{J}}(\varpi_\theta)$ is the least element of $(\breve{\mathfrak{J}}, \Gamma)$. Then K is called the kernel of the Lq^* q-ROMFS homomorphism $\check{\mathfrak{f}}$ and it is denoted by ker $\check{\mathfrak{f}}$.

Proposition 3.36 *A bijective map $\check{\mathfrak{f}}$ of a L_q* q-ROMFSS($\breve{\mathfrak{z}}$) $(\breve{F}, \Delta)$ onto a $(\breve{\mathfrak{J}}, \Gamma)$ is a lattice ordered q-ROMFS isomorphism iff the map $\check{\mathfrak{f}}$ and $\check{\mathfrak{f}}^{-1}$ are lattice ordered q-ROMFS isotone map.*

Proof By Proposition 3.29, If the map $\check{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is a lattice ordered q-ROMFS isomorphism, then the map is lattice ordered q-ROMFS isotone. Also the map $\check{\mathfrak{f}}^{-1}$:$(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is a lattice ordered q-ROMFS isomorphism, then it is a lattice ordered q-rung orthopair multi-fuzzy soft isotone map.

Conversely, suppose that the map $\check{\mathfrak{f}}$: $(\breve{F}, \Delta) \longrightarrow (\breve{\mathfrak{J}}, \Gamma)$ is a bijective map and the maps $\check{\mathfrak{f}}$ and $\check{\mathfrak{f}}^{-1}$ are lattice ordered q-ROMFS isotone maps. We prove that $\check{\mathfrak{f}}$ is a lattice ordered q-ROMFS homomorphism. We know that the maps $\check{\mathfrak{f}}$ and $\check{\mathfrak{f}}^{-1}$ are lattice ordered q-rung orthopair multi-fuzzy soft isotone maps. Now, $\breve{F}(\rho_1) \dot{\subseteq} \breve{F}(\rho_2) \in$

$(\breve{F}, \Delta)$ iff $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\subseteq} \breve{\mathfrak{f}}(\breve{F}(\rho_2))$ iff $\breve{\mathfrak{J}}(\varpi_1)) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_2)) \in (\breve{\mathfrak{J}}, \Gamma)$. Let $\breve{F}(\rho_3) = \breve{F}(\rho_1) \grave{\vee} \breve{F}(\rho_2)$ then $\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_3)$ and $\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_3)$. We get

$$\breve{\mathfrak{J}}(\varpi_1) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_3) \ \ and \ \ \breve{\mathfrak{J}}(\varpi_2) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_3). \tag{13}$$

Now, $\breve{\mathfrak{J}}(\varpi_3) = \breve{\mathfrak{J}}(\varpi_1) \grave{\vee} \breve{\mathfrak{J}}(\varpi_2)$ then $\breve{\mathfrak{J}}(\varpi_1) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_3)$ and $\breve{\mathfrak{J}}(\varpi_2) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_3))$. If $\breve{\mathfrak{f}}(\breve{F}(\rho_4)) = \breve{\mathfrak{J}}(\varpi_4)$, then

$$\breve{\mathfrak{J}}(\varpi_1) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_4) \Rightarrow \breve{\mathfrak{f}}^{-1}(\breve{\mathfrak{J}}(\varpi_1)) \grave{\subseteq} \breve{\mathfrak{f}}^{-1}(\breve{\mathfrak{J}}(\varpi_4))$$
$$\breve{F}(\rho_1) \grave{\subseteq} \breve{F}(\rho_4)$$

and

$$\breve{\mathfrak{J}}(\varpi_2) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_4) \Rightarrow \breve{\mathfrak{f}}^{-1}(\breve{\mathfrak{J}}(\varpi_2)) \grave{\subseteq} \breve{\mathfrak{f}}^{-1}(\breve{\mathfrak{J}}(\varpi_4))$$
$$\breve{F}(\rho_2) \grave{\subseteq} \breve{F}(\rho_4)$$

we get $\breve{F}(\rho_1)) \grave{\vee} \breve{F}(\rho_2)) \grave{\subseteq} \breve{F}(\rho_4))$ implies $\breve{F}(\rho_3)) \grave{\subseteq} \breve{F}(\rho_4))$. Therefore

$$\breve{F}(\rho_3)) \grave{\subseteq} \breve{F}(\rho_4)) \Rightarrow \breve{\mathfrak{J}}(\varpi_3)) \grave{\subseteq} \breve{\mathfrak{J}}(\varpi_4)) \tag{14}$$

From (13) and (14), we get $\breve{\mathfrak{J}}(\varpi_4) = \breve{\mathfrak{J}}(\varpi_1)) \grave{\vee} \breve{\mathfrak{J}}(\varpi_2))$. Therefore $\breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\vee} \breve{F}(\rho_2)) = \breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\vee} \breve{\mathfrak{f}}(\breve{F}(\rho_2))$. Similarly, we get $\breve{\mathfrak{f}}(\breve{F}(\rho_1) \grave{\wedge} \breve{F}(\rho_2)) = \breve{\mathfrak{f}}(\breve{F}(\rho_1)) \grave{\wedge} \breve{\mathfrak{f}}(\breve{F}(\rho_2))$ Thus $\breve{\mathfrak{f}}$ is a lattice ordered q-ROMFS homomorphism. □

Proposition 3.37 *A L_q* q-ROMFSS($\breve{\mathfrak{z}}$) $(\breve{F}, \Delta)$ is modular iff it has no lattice ordered q-ROMFS sublattice isomorphic to M_5.*

Proposition 3.38 *A L_q* q-ROMFSS($\breve{\mathfrak{z}}$) $(\breve{F}, \Delta)$ is distributive iff it has no Lq^* q-ROMFS sublattice isomorphic to M_5.*

4 Modified L_q* q-ROMFSS- MCDM Using Choice Matrix and Weighted Choice Matrix

This section introduces a new MCDM algorithm based on L_q* q-ROMFSSs, with detailed discussions on their corresponding choice matrix and weighted choice matrix.

Definition 4.1 If $\mathcal{H} = [(\mu^m_{\breve{F}_{ij}}, \nu^m_{\breve{F}_{ij}})]_{s \times t} \in L_q*$ q-ROMFS matrix of dimension m, then the choice matrix and weighted choice matrix of L_q* q-ROMFSS is defined as

- When weights are equal

$$\mathcal{C}(\mathcal{H}) = \Bigg[\Bigg(\frac{1}{t}\sum_{j=1}^{t} max\{(\mu^1_{\breve{F}_{ij}}, \mu^2_{\breve{F}_{ij}}, \ldots, \mu^r_{\breve{F}_{ij}})\}^q, \frac{1}{t}\sum_{j=1}^{t} min\{(\nu^1_{\breve{F}_{ij}}, \nu^2_{\breve{F}_{ij}}, \ldots, \nu^r_{\breve{F}_{ij}})\}^q\Bigg)\Bigg]_{s\times 1}, \forall i \tag{15}$$

- When weights are unequal

$$\mathcal{C}_\omega(\mathcal{H}) = \Bigg[\Bigg(\frac{\sum_{j=1}^{t}\omega_j max\{(\mu^1_{\breve{F}_{ij}}, \cdots, \mu^r_{\breve{F}_{ij}})\}^q}{\sum\omega_j}, \frac{\sum_{j=1}^{t}\omega_j min\{(\nu^1_{\breve{F}_{ij}}, \cdots, \nu^r_{\breve{F}_{ij}})\}^q}{\sum\omega_j}\}\Bigg)\Bigg]_{s\times 1}, \forall i$$
$$and \ \ \omega_j > 0 \ \ represents \ \ the \ \ weights. \tag{16}$$

4.1 Algorithm

Input

- $\mathcal{H} = [(\mu^m_{\breve{F}_{ij}}, \nu^m_{\breve{F}_{ij}})]$: L_q* q-ROMFS decision matrix.
- ω_j: weights for each criterion when weights are unequal.
- q: Parameters defining the degree of fuzziness.

Step 1: Define the problem

1. Identify the alternatives $\breve{\mathfrak{Z}} = \{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \ldots, \breve{\mathfrak{z}}_\mathfrak{s}\}$ and criteria $\Delta = \{\rho_1, \rho_2, \ldots, \rho_t\}$ of dimension m.
2. Assign weights ω_j for each criteria ρ_j of dimension m. Ensure weights are positive and normalized ($\sum \omega_j$=1) for unequal weighting.

Step 2: Construct the L_q* q-ROMFS Matrix

1. For each alternative $\breve{\mathfrak{z}}_i$ under each criterion ρj, record the q-rung orthopair multi-fuzzy values:
   - Multi-Membership $\mu^m_{\breve{F}_{ij}}$
   - Multi-Non-Membership $\nu^m_{\breve{F}_{ij}}$
2. Ensure the q-rung orthopair multi-fuzzy condition holds: $(\mu^m_{\breve{F}_{ij}})^q + (\nu^m_{\breve{F}_{ij}})^q \leq 1$

Step 3: Calculate the Choice Matrix

1. Case 1: Equal Weights
   If all criteria are equally important, calculate the choice matrix by averaging the membership and non-membership values for each alternative.
2. Case 2: Unequal Weights
   If the criteria have different importance, calculate the weighted choice matrix, adjusting the values by the criteria weights

Step 4: Select the Optimal Alternative

1. Choose the alternative with the highest membership value from the choice matrix (or weighted choice matrix), as it best matches the criteria.
2. Resolve ties using non-membership values, preferring lower values.

Output
Optimal alternatives and their rankings.

4.2 Selection of Optimal Urban Location for a Green Building Project Using L_q* q-ROMFSS-MCDM

Multiple Criteria Decision-Making (MCDM) is a powerful tool for addressing complex decision-making problems that involve evaluating multiple conflicting criteria. In the context of urban planning, MCDM plays a crucial role in selecting the optimal location for a green building project, ensuring that the chosen site aligns with sustainability objectives. Green building projects demand careful consideration of factors such as solar energy potential, air quality, proximity to public transportation, access to renewable resources, and environmental impact. By applying MCDM techniques, planners can systematically analyze and prioritize these criteria to identify the most suitable

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location. This structured approach not only balances environmental, economic, and social dimensions but also incorporates uncertainty and variability in the decision-making process, ensuring that the final selection contributes to a sustainable and resilient urban future.

4.3 Example

An environmental consultancy is evaluating potential urban sites for a green building project to maximize sustainability and energy efficiency. The aim is to select the most suitable location among four designated sites, $\{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3, \breve{\mathfrak{z}}_4\}$ using the Multiple Criteria Decision-Making (MCDM) approach. Each site is evaluated based on key attributes that influence the project's overall sustainability (see Fig. 6).

Attributes and its three-dimensional evaluation

- Solar Exposure(ρ_1)
  1. High sunlight hours: ideal for implementing rooftop solar panels, ensuring year-round energy generation.
  2. Moderate sunlight hours: suitable for solar energy systems but with limited capacity.
  3. Low sunlight hours: inefficient for solar energy; supplemental energy sources required.
- Air Quality (ρ_2)
  1. Low pollution: excellent for energy-efficient systems requiring clean airflow.
  2. Moderate pollution: Manageable with air filtration systems but less desirable.
  3. High pollution: Poor air quality demands costly mitigation measures.
- Proximity to public transport (ρ_3)
  1. Excellent access: directly supports sustainable urban mobility goals.
  2. Moderate access: encourages mixed commuting methods
  3. Poor access: challenges sustainability objectives due to reliance on private vehicles.
- Water resource availability (ρ_4)
  1. Abundant supply: supports efficient water recycling and conservation initiatives.
  2. Moderate supply: requires water storage systems for consistent availability.
  3. Limited supply: unsustainable without external water sources.

4.4 Illustration

Step 1 Consider the alternatives $\breve{\mathfrak{Z}}$ =$\{\breve{\mathfrak{z}}_1, \breve{\mathfrak{z}}_2, \breve{\mathfrak{z}}_3, \breve{\mathfrak{z}}_4\}$ and Δ =$\{\rho_1, \rho_2, \rho_3, \rho_4\}$ of dimension 3. The weights $\omega = (0.1, 0.2, 0.3, 0.4)^T$ are given for the criteria ρ_1: Soil exposure; ρ_2: Air quality; ρ3: Proximity to public transport; ρ_4: Water resource availability. Each criteria are ordered hierarchically to prioritize system efficiency, reliability, and sustainability, using the L_q* q-ROMFS set information. The criteria are ordered as follows: $\rho_1 \leq \rho_2 \leq \rho_3 \leq \rho_4$ of dimensions (high, medium, low).

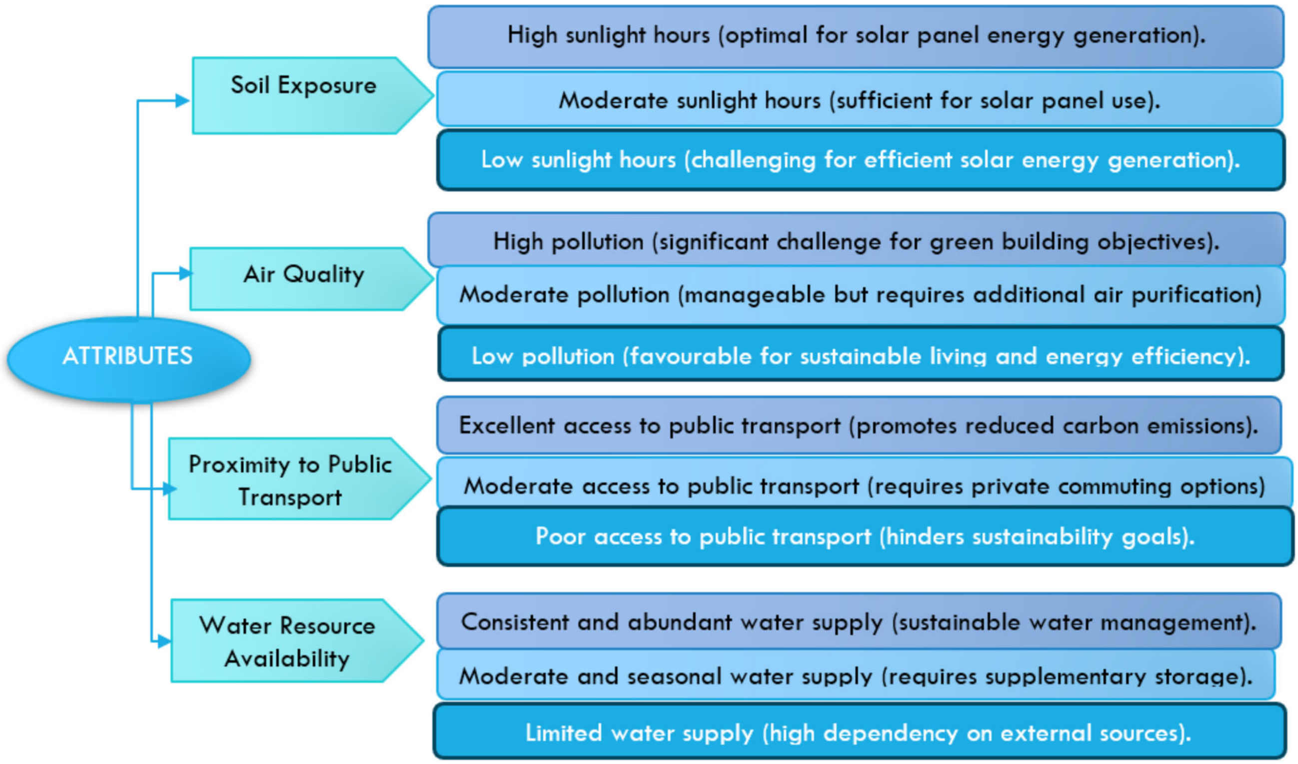

Fig. 6 Multi-dimensional evaluation of each attribute

Step 2. Construct L_q* q-ROMFS matrix (for q = 3) is shown below.

$$\begin{pmatrix} [(0.2,0.8),(0.3,0.6),(0.4,0.5)] & [(0.3,0.7),(0.4,0.7),(0.5,0.4)] & [(0.4,0.5),(0.5,0.4),(0.7,0.4)] & [(0.5,0.4),(0.6,0.4),(0.8,0.2)] \\ [(0.2,0.9),(0.4,0.6),(0.4,0.5)] & [(0.4,0.7),(0.5,0.5),(0.7,0.5)] & [(0.4,0.8),(0.5,0.6),(0.8,0.4)] & [(0.7,0.7),(0.8,0.6),(0.8,0.4)] \\ [(0.3,0.9),(0.4,0.7),(0.5,0.7)] & [(0.4,0.8),(0.5,0.7),(0.6,0.4)] & [(0.5,0.7),(0.7,0.6),(0.8,0.4)] & [(0.6,0.5),(0.7,0.5),(0.9,0.2)] \\ [(0.3,0.8),(0.4,0.5),(0.5,0.4)] & [(0.4,0.7),(0.5,0.6),(0.6,0.5)] & [(0.5,0.6),(0.6,0.4),(0.7,0.3)] & [(0.6,0.5),(0.7,0.4),(0.8,0.3)] \end{pmatrix}_{4\times 3}$$

Step 3.

1. **Case:1** Calculate the choice matrix using Eq. (15). $\mathcal{C}(\mathcal{H}) = \begin{bmatrix} 0.261, 0.06525 \\ 0.358, 0.0945 \\ 0.396, 0.11975 \\ 0.299, 0.06075 \end{bmatrix}$
2. **Case:2** Calculate the weighted choice matrix using Eq. (16).

$$\mathcal{C}_{\omega}(\mathcal{H}) = \begin{bmatrix} 0.3391, 0.0477 \\ 0.4334, 0.0823 \\ 0.5009, 0.0695 \\ 0.3634, 0.0503 \end{bmatrix}$$

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Table 1 Robustness analysis (for $q = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$)

Proposed	$\breve{\mathfrak{z}}_1$	$\breve{\mathfrak{z}}_2$	$\breve{\mathfrak{z}}_3$	$\breve{\mathfrak{z}}_4$	Ranking order
$q = 1$	(0.15134, 0.08587)	(0.5735, 0.053584)	(0.85854, 0.18824)	(0.35019, 0.0827)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$
$q = 2$	(0.14074, 0.05976)	(0.51421, 0.09252)	(0.72884, 0.01503)	(0.46945, 0.07272)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$
$q = 3$	(0.261, 0.06525)	(0.358, 0.0945)	(0.396, 0.11975)	(0.299, 0.06075)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$
$q = 4$	(0.25185, 0.08401)	(0.73638, 0.0943)	(0.82694, 0.15238)	(0.58001, 0.08071)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$
$q = 5$	(0.18587, 0.01618)	(0.56104, 0.08443)	(0.76034, 0.17926)	(0.3615, 0.04496)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$
$q = 6$	(0.19059, 0.04154)	(0.45018, 0.09556)	(0.81389, 0.19454)	(0.7954, 0.07715)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_1$
$q = 7$	(0.237, 0.0673)	(0.802, 0.0781)	(0.997, 0.155)	(0.910, 0.097)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_1$
$q = 8$	(0.17644, 0.02641)	(0.42176, 0.03422)	(0.81817, 0.17285)	(0.6613, 0.04238)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_1$
$q = 9$	(0.22068, 0.07516)	(0.66655, 0.05707)	(0.9538, 0.13197)	(0.7821, 0.1339)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_1$
$q = 10$	(0.12511, 0.0228)	(0.4344, 0.15891)	(0.96573, 0.2556)	(0.5868, 0.0677)	$\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_1$

Step 4.
Ranking of alternatives $\breve{\mathfrak{z}}_3 \geq \breve{\mathfrak{z}}_2 \geq \breve{\mathfrak{z}}_4 \geq \breve{\mathfrak{z}}_1$.
Therefore, $\breve{\mathfrak{z}}_3$ is the most suitable site for a green building project to maximize sustainability and energy efficiency.

5 Robustness Analysis

The robustness analysis presented in Table 1 examines the stability and consistency of alternative rankings across various values of q within the proposed methodology. For $q = 1, 2, 3, 4, 5$, the rankings consistently designate $\breve{\mathfrak{z}}_3$ as the optimal choice, followed by $\breve{\mathfrak{z}}_2$, $\breve{\mathfrak{z}}_4$, and $\breve{\mathfrak{z}}_1$ in that order. When $q = 6, 7, 8, 9, 10$, $\breve{\mathfrak{z}}_3$ remains the optimal alternative, although slight variations occur in the rankings of the other alternatives. These findings demonstrate that the optimal choice is unaffected by changes in the parameter q, affirming the robustness of the proposed methodology. The stability of rankings across different q-values underscores the reliability of this approach in decision-making, even when parameters are adjusted.

5.1 Advantages

The proposed L_q* q-ROMFSS- MCDM methodology offers advantages over previously established MCDM techniques for handling multi-dimensional and lattice-ordered fuzzy information.

- Existing methods such as Lattice ordered SS [2], Lattice ordered FSS [7] and Lattice ordered IFSS [33] fail to manage both membership and non-membership in multi-dimensional scenario. The proposed approach accommodates these complexities and ensures a more robust handling of such cases.
- The MCDM methods used in Lattice Ordered MFSS [39] utilize lattice ordering and a multi-membership function to address dataset vagueness, effectively bridging the gap between truth and falsehood. However, their inability to incorporate a multi-non-membership function limits their effectiveness. The proposed approach addresses this limitation, offering a more comprehensive solution.

5.2 Limitations

- The proposed approach may encounter challenges when both the multi-membership and multi-non-membership functions reach extreme values (e.g., both equal to 1). Such scenarios can introduce ambiguities in decision-making and reduce the effectiveness of the ranking system.

Table 2 Advantages and limitations

Methodology	Advantages	Limitations
Lattice ordered SS [2]	Lattice ordering provides a structured approach to prioritize parameters, ensuring consistency in uncertain scenarios	It cannot represent the multi-dimensional information of parameters
Lattice Ordered FSS [7]	It enables the representation of relationships between objects and attributes in a fuzzy environment using lattice ordering	It is not well-suited for processing data involving non-membership values for the parameters
Lattice Ordered IFSS [33]	Combines membership and non-membership functions with lattice ordering for nuanced decision-making	The sum of membership and non-membership values is restricted to 1
Lattice Ordered MFSS [12]	Lattice ordering of parameters helps handle dataset vagueness using a multi-membership function between truth and falsehood	The lack of a multi-non-membership function restricts the representation of objects in the dataset
L_q* q-ROMFS sets [40]	Leveraging lattice-ordered, multi-dimensional parametric information, the L_q* q-ROMFS set surpasses theories like soft sets, fuzzy soft sets, and intuitionistic fuzzy soft sets	This technique struggles to handle uncertain information where both multi-membership function and multi-non-membership function are equal to 1

- Although the methodology is robust across a range of q, its performance may vary with extreme adjustments in q. Identifying the optimal q value for a given scenario can be non-trivial and might require extensive tuning.
- Lattice ordering provides a structured framework to rank and prioritize parameters effectively. This is especially useful in scenarios involving conflicting or uncertain information, as it ensures consistency and clarity in decision-making. See Table 2 for the advantages and limitations of the proposed methodology.

6 Conclusion and Future Directions

The proposed method effectively handles multi-dimensional decision-making scenarios by combining q-rung orthopair fuzzy sets and multi-fuzzy soft sets with lattice-ordered parameters. By developing algebraic structures such as modularity, distributivity, and homomorphism, the theoretical framework is strengthened. The introduction of choice and weighted choice matrices also offers practical tools for ranking alternatives. This framework, based on L_q* q-ROMFS sets, has been applied to urban location selection for green building projects, taking into account the multi-MDs and multi-NMDs. The concept of raising M-MDs and M-NMDs to the power of q adds flexibility, providing a robust foundation for modeling fuzzy systems and enhancing decision-making under uncertainty. This method bridges the gap between existing structural models and the broader domain of M-MDs and M-NMDs, making it more versatile. Additionally, increasing the value of q expands the decision-making space, allowing for a more comprehensive representation of fuzzy data. A case study on the selection of urban locations for green building projects demonstrates the practical application of the proposed MCDM methods, confirming the approach's effectiveness and stability in addressing complex, multi-criteria group decision-making challenges.
Our future research endeavors focus on several key objectives:

- Expanding Application Domains: We aim to extend the applicability of the proposed methodologies to address a broader range of Multi-Criteria Decision-Making (MCDM) problems. Potential areas include optimizing solutions for challenges like electronic waste management [50], selecting renewable energy sources [47], and other critical decision-making scenarios across diverse sectors.
- Exploration of Advanced Theoretical Models: To improve the capability of managing multi-dimensional and fuzzy information, we plan to delve into advanced hybrid models. This exploration will involve studying and integrating frameworks such as complex q-rung orthopair fuzzy sets, picture fuzzy sets, linear diophantine

fuzzy sets, and bipolar fuzzy sets, which hold significant potential for addressing complex uncertainties in real-world applications.

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References

1. Al-Qudah, Y., Hassan, N.: Complex multi-fuzzy soft set: its entropy and similarity measure. IEEE Access **6**, 65002–17 (2018)
2. Ali, M.I., Mehmood, T., Rehman, M.M., Aslam, M.F.: On lattice ordered soft set. Appl. Soft Comput. **36**, 499–505 (2015)
3. Akram, M., Ali, G., Alcantud, J.C.R.: A novel group decision-making framework under Pythagorean fuzzy N-soft expert knowledge. Eng. Appl. Artif. Intell. **120**, 105879 (2024). https://doi.org/10.1016/j.engappai.2023.105879
4. Arockia Reeta, J., Vimala, J.: A study on distributive and modular lattice ordered fuzzy soft group and its duality. Appl. Math. J. Chin. Univ. **31**, 491–502 (2016)
5. Arshad, M., Saeed, M., Rahman, A.U., Mohammed, M.A., Abdulkareem, K.H., Nedoma, J., et al.: A robust framework for the selection of optimal COVID-19 mask based on aggregations of interval-valued multi-fuzzy hypersoft sets. Expert Syst. Appl. **238**, 121944 (2024)
6. Ashraf, S., Garg, H., Kousar, M.: An industrial disaster emergency decision-making based on China's Tianjin city port explosion under complex probabilistic hesitant fuzzy soft environment. Eng. Appl. Artif. Intell. **123**(1), 106400 (2023). https://doi.org/10.1016/j.engappai.2023.106400
7. Aslam. M.F.: Study of fuzzy soft sets with some order on set of parameters. MS Thesis IIUI (2014)
8. Ali, Z.: Fairly aggregation operators based on complex p, q-Rung orthopair fuzzy sets and their application in decision-making problems. Spectr. Oper. Res. **2**(1), 113–131 (2025). https://doi.org/10.31181/sor21202514
9. Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**(1), 87–96 (1986)
10. Bansal, S., Biswas, S., Singh, S.K.: Fuzzy decision approach for selection of most suitable construction method of Green Buildings. Int. J. Sustain. Built Environ. **6**(1), 122–32 (2017). https://doi.org/10.1016/j.ijsbe.2017.02.005

11. Begam, S.S., Vimala, J., Preethi, D.: A novel study on the algebraic applications of special class of lattice ordered multi-fuzzy soft sets. J. Discrete Math. Sci. Cryptogr. **22**(5), 883–899 (2019). https://doi.org/10.1080/09720529.2019.1685239
12. Begam, S.S., Vimala, J.: Application of lattice ordered multi-fuzzy soft set in forecasting process. J. Intell. Fuzzy Syst. **36**(3), 2323–2331 (2019)
13. Begam, S.S., Vimala, J.: Compositions on lattice ordered multi-fuzzy soft matrix and its simulated application in medical diagnosis. Math. Comput. Simul. **201**, 555–563 (2022)
14. Birkhoff, G.: Lattice Theory. American Mathematical Society, Providence (1967)
15. Biswas, A., Gazi, K.H., Mondal, S.P.: Finding effective factor for circular economy using uncertain MCDM approach. Manag. Sci. Adv. **1**(1), 31–52 (2024). https://doi.org/10.31181/msa1120245
16. Bozanic, D., Epler, I., Puska, A., Biswas, S., Marinkovic, D., Koprivica, S.: Application of the DIBR II—rough MABAC decision-making model for ranking methods and techniques of lean organization systems management in the process of technical maintenance. Facta Univ. Ser. Mech. Eng. **22**(1), 101–123 (2024). https://doi.org/10.22190/FUME230614026B
17. Das, S., Kar, S.: Intuitionistic multi fuzzy soft set and its application in decision making. In: International Conference on Pattern Recognition and Machine Intelligence, pp. 587–592. Springer, Berlin, Heidelberg (2013)
18. Das, A.K., Granados, C.: FP-intuitionistic multi fuzzy N-soft set and its induced FP-Hesitant N-soft set in decision-making. Decis. Mak. Appl. Manag. Eng. **5**(1), 67–89 (2022). https://doi.org/10.31181/dmame181221045d
19. Dey, A., Pal, M.: Generalised multi-fuzzy soft set and its application in decision making. Pac. Sci. Rev. A Nat. Sci. Eng. **17**(1), 23–8 (2015)
20. Gwak, J., Lee, B., Lee, W., Sohn, S.: Optimal location selection for the installation of urban green roofs considering honeybee habitats along with socio-economic and environmental effects. J. Environ. Manag. **189**, 125–33 (2017). https://doi.org/10.1016/j.jenvman.2016.12.022
21. Hussain, A., Ali, M.I., Mahmood, T., Munir, M.: q-Rung orthopair fuzzy soft average aggregation operators and their application in multicriteria decision-making. Int. J. Intell. Syst. **35**(4), 571–599 (2020)
22. Hamid, Ch.T., Abid, M., Naeem, K., Zulqarnain, R. M.: Correlation measures for q-rung orthopair m-polar fuzzy sets with application to pattern recognition. Decis. Mak. Adv. **3**(1), 139–163 (2025). https://doi.org/10.31181/dma31202560
23. Ilamathi, V.S.A., Vimala, J.: Multi-criteria decision making on lattice ordered multisets. In: Thampi, S., Mitra, S., Mukhopadhyay, J., Li, K.C., James, A., Berretti, S. (eds.) Intelligent Systems Technologies and Applications: ISTA 2017, pp. 419–428. Springer, Cham (2018)
24. Imran, R., Ullah, K., Ali, Z., Akram, M.: A multi-criteria group decision-making approach for robot selection using interval-valued intuitionistic fuzzy information and Aczel-Alsina Bonferroni means. Spectr. Decis. Mak. Appl. **1**(1), 1–32 (2024). https://doi.org/10.31181/sdmap1120241
25. Javeed, S., Javed, M., Shafique, I., Shoaib, M., Khan, M.S., Cui, L., et al.: Complex q-rung orthopair fuzzy Yager aggregation operators and their application to evaluate the best medical manufacturer. Appl. Soft Comput. **157**, 111532 (2024). https://doi.org/10.1016/j.asoc.2024.111532
26. Kannan, J., Jayakumar, V., Pethaperumal, M.: Advanced fuzzy-based decision-making: the linear diophantine fuzzy CODAS method for logistic specialist selection. Spectr. Oper. Res. **2**(1), 41–60 (2025). https://doi.org/10.31181/sor2120259
27. Kavitha, S., Janani, K., Mohanrasu, S.S., Satheeshkumar, J., Amudha, T., Rakkiyappan, R.: Ensemble feature selection using q-rung orthopair hesitant fuzzy Hamacher, Einstein and Dombi aggregation operators. Appl. Soft Comput. **161**, 111752 (2024)
28. Kannan, J., Jayakumar, V., Saeid, A.B.: Lattice algebraic structures on LDMFS domains. New Math. Nat. Comput. **2024**, 1–21 (2024). https://doi.org/10.1142/S1793005725500218
29. Karaaslan, F., Cagman, N., Enginoglu, S.: Soft lattices. J New Results Sci. **1**(1), 5–17 (2012)
30. Karaaslan, F., Cagman, N.: Fuzzy soft lattice theory. ARPN J. Sci. Technol. **3**, 2225–7217 (2013)
31. Khan, M.B., Mahmood, T., Iftikhar, M.: Some results on lattice (anti-lattice) ordered double framed soft sets. J New Theory. **29**, 58–70 (2019)
32. Li, Z., Xing, Y., Dong, P.: A novel q-rung orthopair fuzzy best-worst method, Shannon entropy, and MARCOS method for mobile medical app service quality evaluation. Appl. Soft Comput. **155**, 111417 (2024). https://doi.org/10.1016/j.asoc.2024.111417
33. Mahmood, T., Ali, M.I., Malik, M.A., Ahmed, W.: On lattice ordered intuitionistic fuzzy soft set. Int. J. Algebra Stat. **7**(1–2), 46–61 (2018). https://doi.org/10.20454/ijas.2018.1434
34. Maji, P.K., Biswas, R., Roy, A.R.: Fuzzy soft sets. J. Fuzzy Math. **9**(3), 589–602 (2001)
35. Maji, P.K., Roy, A.R., Biswas, R.: On intuitionistic fuzzy soft sets. J. Fuzzy Math. **12**(3), 669–84 (2004)
36. Molodtsov, D.: Soft set theory-first results. Comput. Math. Appl. **37**, 19–31 (1999). https://doi.org/10.1155/2021/6634991
37. Nagarajan, E.K., Geetha, P.: Characterization theorems for modular and distributive soft lattices. J. Innov. Res. Sci. Eng. Technol. **2**, 2319–8753 (2013)
38. Nesticò, A., Passaro, R., Maselli, G., Somma, P.: Multi-criteria methods for the optimal localization of urban green areas. J. Clean. Prod. **374**, 133690 (2022). https://doi.org/10.1016/j.jclepro.2022.133690
39. Peng, X., Yang, Y., Song, J.: Pythagorean fuzzy soft set and its application. Comput. Eng. **41**(7), 224–229 (2015)

40. Pethaperumal, M., Jayakumar, V., Edalatpanah, S.A., Mohideen, A.B.K., Annamalai, S.: An enhanced MADM with L_{q^*} q-ROMFS set in healthcare supplier selection. J. Intell. Fuzzy Syst. Preprint, 1–12 (2024)
41. Pethaperumal, M., Jayakumar, V., Jeevitha, K., Nithya Sri, S.: Selection of healthcare waste disposal strategies with L_{q^*} q-ROMFS set similarity measures. J. Intell. Fuzzy Syst. Preprint, 1–13 (2024)
42. Rahim, M., Shah, K., Abdeljawad, T., Aphane, M., Alburaikan, A., Khalifa, H.A.E.W.: Confidence levels-based p, q-quasirung orthopair fuzzy operators and its applications to criteria group decision making problems. IEEE Access **11**, 109983–109996 (2023)
43. Rahim, M., Ahmad, S., Younis, B.A., Egami, R.H., Ahmed, M.M.: Multiple attribute group decision making based on p, q-quasirung orthopair Bonferroni mean operators and their applications. Evol. Syst. **16**(1), 9 (2025)
44. Saafi, N., Dhouib, K.: An ontological model to enhance traffic conditions in smart city domain. Spectr. Eng. Manag. Sci. **2**(1), 70–84 (2024). https://doi.org/10.31181/sems1120246m
45. Sahoo, S.K., Goswami, S.S.: Green supplier selection using MCDM: a comprehensive review of recent studies. Spectr. Eng. Manag. Sci. **2**(1), 1–16 (2024). https://doi.org/10.31181/sems1120241a
46. Saqlain, M., Saeed, M.: From ambiguity to clarity: unraveling the power of similarity measures in multi-polar interval-valued intuitionistic fuzzy soft sets. Decis. Mak. Adv. **2**(1), 48–59 (2024). https://doi.org/10.31181/dma21202421
47. Seikh, M.R., Chatterjee, P.: Determination of best renewable energy sources in India using SWARA-ARAS in confidence level based interval-valued Fermatean fuzzy environment. Appl. Soft Comput. **155**, 111495 (2024). https://doi.org/10.1016/j.asoc.2024.111495
48. Seikh, M.R., Mandal, U.: q-Rung orthopair fuzzy Archimedean aggregation operators: application in the site selection for software operating units. Symmetry **15**(9), 1680 (2023)
49. Seikh, M.R., Mandal, U.: Q-rung orthopair fuzzy Frank aggregation operators and its application in multiple attribute decision-making with unknown attribute weights. Granul. Comput. **7**, 709–730 (2022). https://doi.org/10.1007/s41066-021-00290-2
50. Seikh, M.R., Chatterjee, P.: Identifying sustainable strategies for electronic waste management utilizing confidence-based group decision-making method in interval-valued Fermatean fuzzy environment. Eng. Appl. Artif. Intell. **135**, 108701 (2024). https://doi.org/10.1016/j.engappai.2024.108701
51. Sebastian, S., Ramakrishnan, T.V.: Multi-fuzzy sets: an extension of fuzzy sets. Fuzzy Inf. Eng. **1**, 35–43 (2011)
52. Sebastian, S., Ramakrishnan, T.V.: Multi-fuzzy sets. Int. Math. Forum **5**, 2471–2476 (2010)
53. Vimala, J., Mahalakshmi, P., Rahman, A.U., Saeed, M.: A customized TOPSIS method to rank the best airlines to fly during COVID-19 pandemic with q-ROMFS information. Soft. Comput. **27**(20), 14571–14584 (2023). https://doi.org/10.1007/s00500-023-08976-2
54. Vimala, J., Rajareega, S., Preethi, D., Madeleine, A.T.: Lattice ordered neutrosophic soft set and its usage for a large-scale company in adopting best NGO to deliver funds during COVID-19. J. Comput. Cogn. Eng. **2**(3), 210–219 (2022). https://doi.org/10.47852/bon-viewJCCE2202193
55. Vimala, J., Garg, H., Jeevitha, K.: Prognostication of myocardial infarction using lattice ordered linear diophantine multi-fuzzy soft set. Int. J. Fuzzy Syst. **26**, 44–59 (2024). https://doi.org/10.1007/s40815-023-01574-2
56. Wang, P., Zhu, B., Yu, Y., Ali, Z., Almohsen, B.: Complex intuitionistic fuzzy DOMBI prioritized aggregation operators and their application for resilient green supplier selection. Facta Univ. Ser. Mech. Eng. **21**(3), 339–357 (2023). https://doi.org/10.22190/FUME230805029W
57. Yager, R.R.: On the theory of bags. Int. J. Gen. Syst. **13**(1), 23–37 (1986)
58. Yager, R.R.: Pythagorean fuzzy subsets. In: Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), pp. 57–61 (2013)
59. Yager, R.R.: Generalized orthopair fuzzy sets. IEEE Trans. Fuzzy Syst. **25**(5), 1222–1230 (2017). https://doi.org/10.1109/TFUZZ.2016.2604005
60. Yang, Y., Tan, X., Meng, C.: The multi-fuzzy soft set and its application in decision making. Appl. Math. Model. **37**(7), 4915–4923 (2013)
61. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
62. Zhang, H.D., Shu, L.: Possibility multi-fuzzy soft set and its application in decision making. J. Intell. Fuzzy Syst. **27**(4), 2115–2125 (2014)
63. Zulqarnai, R.M., Naveed, H., Siddique, I., Alcantud, J.C.R.: Transportation decisions in supply chain management using interval-valued q-rung orthopair fuzzy soft information. Eng. Appl. Artif. Intell. **133**, 108410 (2024)

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