Complex Variables, Theory and Application: An International Journal

ISSN: 0278-1077 (Print) 1563-5066 (Online) Journal homepage: www.tandfonline.com/journals/gcov19

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Daniyal M. Israfilov

To cite this article: Daniyal M. Israfilov (2001) Faber series in weighted bergman spaces, Complex Variables, Theory and Application: An International Journal, 45:2, 167-181, DOI: 10.1080/17476930108815375

To link to this article: https://doi.org/10.1080/17476930108815375

Published online: 29 May 2007.

Complex Variables, Vol 45, pp. 167–181
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Faber Series in Weighted Bergman Spaces

DANIYAL M. ISRAFILOV*

Department of Mathematics, Balikesir University, 10100 Balikesir, Turkey

Communicated by H. Begehr

(Received 3 February 2000; In final form 20 August 2000)

Let $G\subset C$ be a finite domain with a quasiconformal boundary L and let $A^p(G,\omega)$ be a weighted Bergman space of analytic functions in G. In this work, the generalized Faber series for the functions in $A^p(G,\omega)$ are defined and its approximation properties are investigated.

Keywords: Generalized Faber polynomials; Generalized Faber series; Approximation properties; Quasiconformal boundary; Weighted Bergman spaces

AMS Subject Classification Numbers: 30E10, 41A10, 41A25, 41A58

1. INTRODUCTION

Let $E\subset C$ be a finite continuum with more than one point whose complement is connected and $D:=\{\omega:|\omega|<1\}$. We denote by $\omega=\varphi(z)$ the conformal mapping of $CE:=\overline{C}\backslash E$ onto $C\overline{D}:=\overline{C}\backslash\overline{D}$ with normalization $\varphi(\infty)=\infty$, $\varphi'(\infty)>0$, where $\overline{C}$ and $\overline{D}$ are the closures of the complex plane C and D respectively. The inverse mapping we denote by $\psi=\varphi^{-1}$. For an arbitrary fixed number $R>1$ we put

$$L_R:=\{z:|\varphi(z)|=R\},\quad E_R:=\{z:z\in CE,|\varphi(z)|<R\}\cup E.$$

*Corresponding author. e-mail: mdaniyal@mail.balikesir.edu.tr

Let g be an analytic function in CE, $g(\infty) > 0$. As known ([11], p. 60), the generalized Faber polynomials $F_n(z, g)$, $n = 1, 2, \ldots$ associated with E and g, are defined through the expansion

$$\frac{\omega g[\psi(\omega)]\psi'(\omega)}{\psi(\omega) - z} = \sum_{n=0}^{\infty} \frac{F_n(z, g)}{\omega^n}, \tag{1}$$

which converges uniformly and absolutely on compact subsets of $\{C\overline{D}, CE\}$. Differentiation of (1) with respect to z gives

$$\frac{\omega g[\psi(\omega)]\psi'(\omega)}{[\psi(\omega) - z]^2} = \sum_{n=1}^{\infty} \frac{F_n'(z, g)}{\omega^n}, \quad z \in E, \ |\omega| > 1. \tag{2}$$

It is easy to verify that, for every natural number n, $F_n(z, g)$ is a polynomial of degree n.

Let G be a finite domain with a quasiconformal boundary L, $0 \in G$, and let ω be a weight function given on G. We recall that L is a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto L.

For functions f analytic in G we set

$$A^p(G, \omega) := \left\{ f : \int\int_G |f(z)|^p \omega(z) d\sigma_z < \infty \right\}, \quad p > 1,$$

$$A(G) := \left\{ f : \int\int_G |f(z)| d\sigma_z < \infty \right\}.$$

If $\omega = 1$ we denote $A^p(G) := A^p(G, 1)$.

We refer to the spaces $A^p(G, \omega)$ as "weighted Bergman spaces". As known the usual "Bergman spaces" are denoted by $A^p(G)$. $A^p(G, \omega)$ becomes a normed space if we define

$$\|f\|_{A^p(G,\omega)} := \left(\int\int_G |f(z)|^p \omega(z) d\sigma_z \right)^{1/p}.$$

As is known the Faber polynomials and Faber series, and their generalizations have been used to provide polynomial and rational approximations on the domains of the complex plane. Here for the expansion of the functions to the Faber or generalized Faber series were used usually the familiar Cauchy integral representation of analytic functions. This integral representation is correct only on the domains with the rectifiable boundaries. Approximation properties of these series are described in the books of Smirnov and Lebedev [10], Gaier [6] and Suetin [11].

In this work for the expansions of the analytic functions to the generalized Faber series in the domains with a quasiconformal boundary we use the integral representation

$$f(z)=-\frac{1}{\pi}\int\int_{C\overline{G}}\frac{(f\circ y)(\varsigma)}{(\varsigma-z)^2}y_{\bar\varsigma}(\varsigma)d\sigma_\varsigma,\quad z\in G, \tag{3}$$

given by Belyi [4] for the functions f, analytic and bounded on the domain G. Note that the quasiconformal boundaries may be even locally nonrectifiable. Here $y=y(\varsigma)$ is a quasiconformal reflection across the boundary L, and as follows from Ahlfors lemma [1] (see [2], p. 26, Corollary 1.3), can always be chosen canonical in the sense that it is differentiable on C almost everywhere, except possibly at the points of the curve L and for any sufficiently small fixed $\delta>0$ it satisfies the relations

$$\begin{aligned}&|y_\varsigma|+|y_{\bar\varsigma}|\leq c_1,\quad \delta<|\varsigma|<\frac{1}{\delta};\ \ \varsigma\notin L,\\ &|y_\varsigma|+|y_{\bar\varsigma}|\leq c_2|\varsigma|^{-2},\quad |\varsigma|\geq\frac{1}{\delta},\ \ |\varsigma|\leq\delta.\end{aligned} \tag{4}$$

Considering only the canonical quasiconformal reflections. Batchayev [3] improved Belyi's result, having proved that equality (3) is fulfilled if and only if $f\in A(G)$. The accurate proof of the cited Batchaev's result is given in [2] (see [2], p. 110, Th. 4.4). Here and in the following y will be considered as a canonical reflection across the boundary L.

Let $f\subset A(G)$. Denoting $E:=\overline{G}$ in particular, and substituting $\varsigma=\psi(\omega)$ in (3) we get

$$\begin{aligned}f(z)&=-\frac{1}{\pi}\int\int_{C\overline{D}}f(y(\psi(w)))y_{\bar\varsigma}[\psi(w)]\frac{\psi'(w)\overline{\psi'(w)}}{[\psi(w)-z]^2}d\sigma_w\\ &=-\frac{1}{\pi}\int\int_{C\overline{D}}\frac{f(y(\psi(w)))y_{\bar\varsigma}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]}\frac{g[\psi(w)]\psi'(w)}{[\psi(w)-z]^2}d\sigma_w.\end{aligned} \tag{5}$$

From (2) and (5) it follows that

$$f(z) \sim \sum_{m=1}^{\infty} a_m(f,g) F'_m(z,g), \quad z \in G, \tag{6}$$

where

$$a_m(f,g) = -\frac{1}{\pi} \int\int_{C\overline{D}} \frac{f(y(\psi(w))) y_{\bar{\varsigma}}[\psi(w)] \overline{\psi'(w)}}{g[\psi(w)] w^{m+1}} d\sigma_w, \quad m = 1, 2, \ldots$$

Since the series (6) is a derivative series of a generalized Faber series, then for simplicity, we also call this series a generalized Faber series and the coefficients $a_m(f,g)$, $m = 1, 2, \ldots$ generalized Faber coefficients of f.

Let p and q be the conjugate exponents, *i.e.*, $(1/p) + (1/q) = 1$, and let

$$G_{R_0} := \{z : z \in C\overline{G}, |\varphi(z)| < R_0\} \cup \overline{G}, \quad R_0 > 1.$$

Definition 1 Let g be an analytic function in $C\overline{G}$ with $g(\infty) > 0$. We say that $g \in B_p(C\overline{G})$ if

$$\int\int_{G_{R_0} \setminus G} \frac{|\varphi'(\varsigma)|^{2-p}}{|g(\varsigma)|^p} d\sigma_\varsigma \cdot \int\int_{G_{R_0} \setminus G} |g(\varsigma)|^q |\varphi'(\varsigma)|^{2-q} d\sigma_\varsigma < \infty, \tag{7}$$

for some fixed constant $R_0 \in (1, \infty)$.

Definition 2 For every $p > 1$ and $g \in B_p(C\overline{G})$ we define a weight function ω in the following way

$$\omega(z) := \frac{|(\varphi \circ y)_{\bar{z}}|^{2-p}}{|(g \circ y)(z)|^p}, \quad z \in G.$$

As follows from this definition the class of weight functions ω is described in terms of the functions g and φ, and using the quasiconformal reflection y across the boundary L. But in the case of $p = 2$ the weight function ω only depends g and y.

In what follows we shall assume that $p \in (1, \infty)$.

Our new results are summarized in the following theorems.

THEOREM 1 *Let* $f \in A^p(G, \omega)$ *and* $g \in B_p(C\overline{G})$. *Then a generalized Faber series (6) of f converges pointwise to f in G.*

COROLLARY 1 *Let* $g \in B_p(C\overline{G})$. *Then no two different functions in* $A^p(G, \omega)$ *have the same generalized Faber series (6).*

We may also study the uniqueness problem for the generalized Faber series (6).

THEOREM 2 *Let* $g \in B_p(C\overline{G})$ *and nonvanishing in* $C\overline{G}$ *and let* b_m, $m = 1, 2, \ldots$ *be a sequence of complex numbers. If the series* $\sum_{m=1}^{\infty} b_m F_m'(z, g)$ *converges to a function* $f \in A^p(G, \omega)$ *in the norm* $\|\cdot\|_{A^p(G,\omega)}$, *then* b_m, $m = 1, 2, \ldots$, *are the generalized Faber coefficients of f.*

Finally, if $p = 2$ and $S_n(f, g, z) := \sum_{m=1}^{n+1} a_m(f, g) F_m'(z, g)$ is the nth-partial sum of the generalized Faber series of $f \in A^2(E_R, \omega)$ we estimate the error $\|f - S_n(f, g, \cdot)\|_{A^2(E)}$ by $E_n(f, E_R, \omega)$, where

$$E_n(f, E_R, \omega) := \inf\{\|f - P_n\|_{A^2(E_R,\omega)} : P_n \text{ is a polynomial of degree } \leq n\},$$

denotes the minimal error in approximating f by polynomials of degree at most n.

THEOREM 3 *Let g be a function, analytic and nonvanishing in* CE *and let* $f \in A^2(E_R, \omega)$. *Then for every natural number n and* $r \subset (1, R)$, *we have*

$$\|f - S_n(f, g, \cdot)\|_{A^2(E)} \leq c\left(\frac{r}{R}\right)^n E_n(f, E_R, \omega), \quad R > 1,$$

where

$$c = \frac{\sqrt{\pi} M(g, r) r}{(1 - k_R^2)\sqrt{(R^2 - r^2)(r^2 - 1)}},$$

and $M(g, r) := \frac{1}{2\pi} \int_{|w|=r} |g[\psi(w)]| |dw|$.

If in addition the function $g[\psi(w)]$ belongs to the familiar Hardy space $H^1(CD)$, then we have the following estimation.

THEOREM 4 *Let the conditions of Theorem* 3 *hold and in addition* $g[\psi(w)] \in H^1(CD)$. *Then*

$$\| f - S_n(\cdot, f, g)\|_{A^2(E)} \le c \frac{E_n(f, E_R, \omega)\sqrt{n+2}}{R^n},$$

where

$$c = \frac{\sqrt{\pi} M(g, 1)}{(1 - k_R^2)\sqrt{(R^2 - 1)}}.$$

Note that in the nonweighted case for $p=2$, $E := \overline{G}$ and $R=1$ the results presented here were stated and proved in [8] and in [5] respectively. For $p=2$, Theorems 1 and 2 in the weighted case, and Theorem 4 in the nonweighted case were obtained in [9]. Moreover, in [9] for the case $p=2$ also succeeded in proving uniformly convergence of the generalized Faber series (6) on compact subsets of G.

Similar problems in $A(\overline{G})$, where $A(\overline{G})$ denotes the class of functions which are continuous in $\overline{G}$ and analytic in G, were studied in [7].

We shall use $c, c_1, c_2, \ldots$ to denote constants depending only on numbers that are not important for the questions of our interest.

2. AUXILIARY RESULTS

LEMMA 1 *Let g be an analytic function satisfying the* $B_p(C\overline{G})$ *condition and nonvanishing in* $C\overline{G}$. *Then*

$$a_n(F'_m, g) = \begin{cases} 1, & m = n, \\ 0, & m \ne n. \end{cases}$$

Proof Since $g \in B_p(C\overline{G})$, using the condition (7) it is easy to check that the definition of the coefficients $a_n(F'_m, g)$ is correct for every $n = 1, 2, \ldots,$. Further applying the Green's formulae and the Cauchy's integral theorem we have

$$a_n(F'_m, g) = -\frac{1}{\pi} \int\int_{C\overline{D}} \frac{F'_m(y(\psi(w)), g) y_{\bar\zeta}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)] w^{n+1}} d\sigma_w$$

$$= -\frac{1}{\pi}\lim_{\substack{r\to 1+,\\ R\to\infty}} \int\int_{r<|w|<R} \frac{F'_m(y(\psi(w)),g)y_{\bar\varsigma}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]w^{n+1}}d\sigma_w$$

$$= -\frac{1}{\pi}\lim_{\substack{r\to 1+,\\ R\to\infty}} \int\int_{r<|w|<R} \frac{d}{d\bar w}\frac{F_m(y(\psi(w)),g)}{g[\psi(w)]w^{n+1}}d\sigma_w$$

$$= -\frac{1}{2\pi i}\lim_{R\to\infty}\int_{|w|=R}\frac{F_m[y(\psi(w)),g]}{g[\psi(w)]w^{n+1}}dw + \frac{1}{2\pi i}\lim_{r\to 1+}\int_{|w|=r}\frac{F_m[y(\psi(w)),g]}{g[\psi(w)]w^{n+1}}dw$$

$$= 0 + \frac{1}{2\pi i}\int_{|w|=1}\frac{F_m[\psi(w),g]}{g[\psi(w)]w^{n+1}}dw$$

$$= \frac{1}{2\pi i}\int_{|w|=r>1}\frac{F_m[\psi(w),g]}{g[\psi(w)]w^{n+1}}dw \tag{8}$$

Since by [11],

$$F_m(z,g) = g(z)\varphi^m(z) + E_m(z,g), \quad z\in C\overline{G}, \tag{9}$$

where $E_m(z,g)$ is analytic in $C\overline{G}$ and $E_m(\infty,g)=0$, from (8) we get

$$a_n(F'_m,g) = \frac{1}{2\pi i}\int_{|w|=r>1} w^{m-n-1}dw + \frac{1}{2\pi i}\int_{|w|=r>1}\frac{E_m(\psi(w),g)}{g[\psi(w)]w^{n+1}}dw$$

$$= \frac{1}{2\pi i}\int_{|w|=r>1} w^{m-n-1}dw = \begin{cases}1, & m=n,\\ 0, & m\neq n.\end{cases}$$

and the proof of Lemma 1 is complete.

LEMMA 2 *Let $f\in A^p(G,\omega)$ and y be a K-quasiconformal reflection across the boundary L. Then*

$$\int\int_{C\overline{G}}\left|\frac{(f\circ y)(\varsigma)y_{\bar\varsigma}(\varsigma)}{g(\varsigma)}\right|^p|\varphi'(\varsigma)|^{2-p}d\sigma_\varsigma \le \frac{\|f\|^p_{A^p(G,\omega)}}{(1-k^2)^p}, \quad k=\frac{K-1}{K+1}.$$

Proof Since $\bar y$ is a K-quasiconformal mapping of $\overline{C}$ onto itself, $|y_\varsigma|/|y_{\bar\varsigma}| = |\bar y_{\bar\varsigma}|/|\bar y_\varsigma| \le k$ and $|\bar y_\varsigma|^2 - |\bar y_{\bar\varsigma}|^2 > 0$. Therefore, if

$J_y = |y_\varsigma|^2 - |y_{\bar\varsigma}|^2$ is the Jacobian of y we get

$$\int\int_{C\overline{G}} \left| \frac{(f \circ y)(\varsigma) y_{\bar\varsigma}(\varsigma)}{g(\varsigma)} \right|^p |\varphi'(\varsigma)|^{2-p} d\sigma_\varsigma$$

$$= \int\int_{C\overline{G}} \left| \frac{(f \circ y)(\varsigma) |y_{\bar\varsigma}(\varsigma)|^2 J_y^{-1} J_y}{g(\varsigma) y_{\bar\varsigma}(\varsigma)} \right|^p |\varphi'(\varsigma)|^{2-p} d\sigma_\varsigma$$

$$= \int\int_{C\overline{G}} \left[1 - (|y_\varsigma| / |y_{\bar\varsigma}|)^2\right]^{-p}$$

$$\left| \frac{(f \circ y)(\varsigma) J_y(\varsigma)}{g(\varsigma) y_{\bar\varsigma}(\varsigma)} \right|^p |\varphi'(\varsigma)|^{2-p} d\sigma_\varsigma$$

$$\leq \frac{1}{(1-k^2)^p} \int\int_G \left| \frac{f(\varsigma)}{(g \circ y^{-1})(\varsigma)} \cdot \frac{J_{y^{-1}} \cdot J_y}{(y^{-1})_{\bar\varsigma}} \right|^p$$

$$|\varphi'(y^{-1}(\varsigma))|^{2-p} \cdot J_{y^{-1}} d\sigma_\varsigma$$

$$\leq \frac{1}{(1-k^2)^p} \int\int_G \left| \frac{f(\varsigma)}{(g \circ y^{-1})(\varsigma)} \right|^p |(y^{-1})_{\bar\varsigma}|^{-p}$$

$$|\varphi'(y^{-1}(\varsigma))|^{2-p} |y_{\bar\varsigma}^{-1}|^2 d\sigma_\varsigma$$

$$= \frac{1}{(1-k^2)^p} \int\int_G \left| \frac{f(\varsigma)}{(g \circ y)(\varsigma)} \right|^p$$

$$|(\varphi' \circ y)_{\bar\varsigma}|^{2-p} \, d\sigma_\varsigma = \frac{\| f \|^p_{A^p(G,\omega)}}{(1-k^2)^p}$$

and the proof is complete. For $f = 1$, $\omega = 1$, and $p = 2$ this lemma was proved in [4].

LEMMA 3 *Let g be an analytic function in $C\overline{G}$ and $g(\infty) > 0$. Then for every natural number n and $r \in (1, R)$ we have*

$$\sum_{m=n+2}^{\infty} \frac{\|F'_m(\cdot, g)\|^2_{A^2(E)}}{m R^{2m}} \leq c \left(\frac{r}{R} \right)^{2(n+1)},$$

where

$$c = \frac{\pi r^2 [M(g,r)]^2}{(R^2 - r^2)(r^2 - 1)},$$

and $M(g,r) := \frac{1}{2\pi} \int_{|w|=r} |g[\psi(w)]| |dw|$.

Proof The function $g[\psi(w)]$ is analytic in CE and $g[\psi(\infty)] > 0$. Therefore it have the Laurent expansion

$$g[\psi(w)] = \alpha_0 + \frac{a_1}{w} + \dots, \quad |w| > 1, \tag{10}$$

with Laurent coefficients

$$\alpha_k = \frac{1}{2\pi i}\int_{|w|=r} \frac{g[\psi(w)]dw}{w^{-k+1}}, \quad k = 0, 1, 2, \dots, \quad r > 1, \tag{11}$$

which satisfy the relations

$$|\alpha_k| \le r^{k-1} M(g, r), \quad k = 0, 1, 2, \dots, \quad r > 1. \tag{12}$$

On the other hand setting $z = \psi(w)$ in (9) we have

$$F_m[\psi(w), g] = g[\psi(w)]w^m + \sum_{k=1}^{\infty} \frac{\lambda_k^{(m)}}{w^k}, \quad |w| > 1,$$

and from this by (10) we conclude that

$$\begin{aligned} F_m[\psi(w), g] &= \alpha_0 w^m + \alpha_1 w^{m-1} + \dots + \alpha_{m-1} w + \alpha_m \\ &\quad + \sum_{k=1}^{\infty} \frac{\beta_k^{(m)}}{w^k}, \quad |w| > 1. \end{aligned} \tag{13}$$

Let $S_m(E)$ be the area of the image of E under $F_m(z, g)$ in the Riemann surface of F_m. Then by means of a theorem due to Lebedev and Milin (given in [11, p. 170]) and according to (12) from (13) we have

$$\begin{aligned} S_m(E) &= \pi\left(\sum_{k=1}^{m} k|\alpha_k|^2 - \sum_{k=1}^{\infty} k|\beta_k^{(m)}|^2\right) \le \pi \sum_{k=1}^{m} k|\alpha_k|^2 \\ &\le \pi[M(g, r)]^2 \sum_{k=1}^{m} k r^{2k-2} \le \pi m [M(g, r)]^2 \sum_{k=1}^{m} r^{2k-2} \\ &\le \frac{\pi[M(g, r)]^2}{r^2 - 1} m r^{2m}. \end{aligned} \tag{14}$$

On the other hand

$$S_m(E) = \int\int_E |F_m'(z, g)|^2 d\sigma_z = \|F_m'(\cdot, g)\|^2_{A^2(E)}, \tag{15}$$

and from (14) and (15) we finally get

$$\sum_{m=n+2}^{\infty} \frac{\|F'_m(\cdot,g)\|^2_{A^2(E)}}{mR^{2m}} \le \frac{\pi[M(g,r)]^2}{r^2-1} \sum_{m=n+2}^{\infty} \frac{mr^{2m}}{mR^{2m}}$$

$$= \frac{\pi[M(g,r)]^2}{r^2-1} \sum_{m=n+2}^{\infty} \left(\frac{r^2}{R^2}\right)^m$$

$$\le \frac{\pi r^2[M(g,r)]^2}{(R^2-r^2)(r^2-1)} \left(\frac{r}{R}\right)^{2(n+1)}.$$

The proof is complete.

If the function $g[\psi(w)]$ belongs to the familiar Hardy space $H^1(C\overline{D})$ then $|\alpha_k| \le M(g,1)$, $k=0,1,2,\dots$, and from the proof of Lemma 3 we obtain

Corollary 2 *If* $g[\psi(w)] \in H^1(C\overline{D})$ *then*

$$\sum_{m=n+2}^{\infty} \frac{\|F'_m\|^2_{A^2(E)}}{mR^{2m}} \le \frac{\pi[M(g,1)]^2(n+2)R^2}{(R^2-1)^2 R^{2(n+1)}}.$$

Note that in general the above estimations are precise in the sense that, the degree of n in the factor $1/R^{2(n+1)}$ cannot be increase even in the special case $E := \overline{D}$. Indeed for $E = \overline{D}$ and $g = 1$ we have $F_m(z) = z^m$, $M(g,1) = 1$ and

$$\sum_{m=n+2}^{\infty} \frac{\|F'_m\|^2_{A^2(E)}}{mR^{2m}} = \frac{\pi}{(R^2-1)R^{2(n+1)}}.$$

3. PROOF OF THE NEW RESULTS

Proof of Theorem 1 Let $f \in A^p(G,\omega)$ and $g \in B_p(C\overline{G})$. First of all we prove that $f \in A(G)$. Since for any sufficiently small fixed $\delta < 0$, according to the relations (4) and (7) we have

$$\int\int_G |(\varphi \circ y)_{\bar z}|^{2-q} |(g \circ y)(z)|^q d\sigma_z$$

$$= \int\int_{C\overline{G}} |g(z)|^q |y_{\bar z}|^{2-q} (|y_{\bar z}|^2 - |y_z|^2)^{q-1} |\varphi'(z)|^{2-q} d\sigma_z$$

$$\leq \int\int_{C\overline{G}} |\varphi'(z)|^{2-q}|g(z)|^q|y_{\bar{z}}|^{2-q}|y_{\bar{z}}|^{2(q-1)}d\sigma_z$$

$$\leq \int\int_{C\overline{G}} |\varphi'(z)|^{2-q}|g(z)|^q|y_{\bar{z}}|^q d\sigma_z = \int\int_{G_{R_0}\setminus\overline{G}} \cdots + \int\int_{CG_{R_0}} \cdots$$

$$\leq c_1 \int\int_{G_{R_0}\setminus\overline{G}} |\varphi'(z)|^{2-q}|g(z)|^q d\sigma_z + c_3 \int\int_{CG_{R_0}} |y_{\bar{z}}|^q d\sigma_z$$

$$\leq c_1 \int\int_{G_{R_0}\setminus\overline{G}} |\varphi'(z)|^{2-q}|g(z)|^q d\sigma_z$$

$$+ c_3 \int\int_{|z|\geq c(R_0)} |z|^{-2q} d\sigma_z < \infty,$$

where $c_3 = \max\{|g(z)|^q|\varphi'(z)|^{2-q} : z \in CG_{R_0}\}$, applying the Hölder inequality and taking into account the above relation we conclude that

$$\int\int_G |f(z)|d\sigma_z = \left(\int\int_G \left|\frac{f(z)}{(g\circ y)(z)}\right| |(\varphi\circ y)_{\bar{z}}|^{(2-p)/p}\right.$$

$$\left. |(\varphi\circ y)_{\bar{z}}|^{(2-p)/p}|(g\circ y)(z)|d\sigma_z\right)$$

$$\leq \left(\int\int_G |f(z)|^p \frac{|(\varphi\circ y)_{\bar{z}}|^{2-p}}{|(g\circ y)(z)|^p} d\sigma_z\right)^{1/p}$$

$$\left(\int\int_G |(g\circ y)(z)|^q |(\varphi\circ y)_{\bar{z}}|^{2-q} d\sigma_z\right)^{1/q} < \infty.$$

Therefore, $f \in A(G)$ and the representation (3) is correct. Further, for every $z \in G$ by virtue of (2) we have

$$-\frac{1}{\pi}\int\int_{C\overline{D}} \frac{f(y(\psi(w)))y_{\bar{\zeta}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]} \frac{g[\psi(Rw)]\psi'(Rw)}{[\psi(Rw)-z]^2} d\sigma_w$$

$$= \sum_{n=1}^{\infty} R^{-n-1}F_n'(z,g)\left(-\frac{1}{\pi}\right)\int\int_{C\overline{D}} \frac{f(y(\psi(w)))y_{\bar{\zeta}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]w^{n+1}} d\sigma_w$$

$$= \sum_{n=1}^{\infty} R^{-n-1}a_nF_n'(z,g). \tag{16}$$

On the other hand, by Lemma 2,

$$\int\int_{C\overline{D}} \left|\frac{f(y(\psi(w)))y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]}\right|^p d\sigma_w$$
$$\le \int\int_{C\overline{G}} \left|\frac{(f\circ y)(\varsigma)y_{\bar{\varsigma}}(\varsigma)}{g(\varsigma)}\right|^p |\varphi'(\varsigma)|^{2-p} d\sigma_\varsigma \le \frac{\|f\|^p_{A^p(G,\omega)}}{(1-k^2)^p},$$

and

$$\int\int_{C\overline{D}} \left|\frac{g[\psi(Rw)]\psi'(Rw)}{[\psi(Rw)-z]^2}\right|^q d\sigma_w$$
$$= \frac{1}{R^2}\int\int_{|w|>R} \left|\frac{g[\psi(w)]\psi'(w)}{[\psi(w)-z]^2}\right|^q d\sigma_w$$
$$= \frac{1}{R^2}\int\int_{CG_R} \frac{|g(\varsigma)|^q|\varphi'(z)|^{2-q}}{|\varsigma-z|^{2q}} d\sigma_\varsigma$$
$$\le c_4\int\int_{CG_R} \frac{d\sigma_\varsigma}{|\varsigma-z|^{2q}} \le c_4 2\pi \int_{dist(z,\partial G)}^{\infty} r^{1-2q}dr$$
$$= c_4 2\pi \frac{[dist(z,\partial G)]^{2(1-q)}}{2(q-1)} \le c_5 < \infty, \quad z\in G,$$

uniformly with respect to $R \ge 1$. Then from (16) we get

$$f(z) = -\frac{1}{\pi}\int\int_{C\overline{D}} \frac{f(y(\psi(w)))y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]} \frac{g[\psi(w)]\psi'(w)}{[\psi(w)-z]^2} d\sigma_w$$
$$= \lim_{R\to 1+} -\frac{1}{\pi}\int\int_{C\overline{D}} \frac{f(y(\psi(w)))y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]} \frac{g[\psi(Rw)]\psi'(Rw)}{[\psi(Rw)-z]^2} d\sigma_w$$
$$= \sum_{n=1}^{\infty} a_n F_n'(z,g),$$

for every $z\in G$. This completes the proof.

Proof of Theorem 2 Let $S_n(z) := \sum_{m=1}^{n+1} b_m F'_m(z,g)$ be the nth-partial sums of $\sum_{m=1}^{\infty} b_m F'_m(z,g)$. By Lemma 1, we get

$$\lim_{n\to\infty} -\frac{1}{\pi}\int\int_{C\bar{D}} \frac{(S_n \circ y)(\psi(w))y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}}{g[\psi(w)]w^{m+1}} d\sigma_w = b_m, \quad m = 1,2,\ldots \tag{17}$$

On the other hand, by Hölder's inequality and by Lemma 2 we have

$$\begin{aligned}
|a_m(f,g) - b_m| &\le \left|\frac{1}{\pi}\int\int_{C\bar{D}} \frac{(f\circ y)(\psi(w)) - (S_n\circ y)(\psi(w))}{g[\psi(w)]w^{m+1}}\right. \\
&\qquad \left. y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}d\sigma_w\right| \\
&\quad + \left|-\frac{1}{\pi}\int\int_{C\bar{D}} \frac{(S_n\circ y)(\psi(w))}{g[\psi(w)]w^{m+1}} y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}d\sigma_w - b_m\right| \\
&\le c_6\left(\int\int_{C\bar{G}} \left|\frac{f(y(\varsigma)) - S_n(y(\varsigma))}{g(\varsigma)}\right|^p |\varphi'(\varsigma)|^{2-p} d\sigma_\varsigma\right)^{1/p} \\
&\qquad \left(\int\int_{C\bar{D}} \frac{d\sigma_w}{|w|^{q(m+1)}}\right)^{1/q} \\
&\quad + \left|-\frac{1}{\pi}\int\int_{C\bar{D}} \frac{(S_n\circ y)(\psi(w))}{g[\psi(w)]w^{m+1}} y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}d\sigma_w - b_m\right| \\
&\le c_7\left(\frac{2}{2-q(m+1)}\right)^{1/q} \|f - S_n\|_{A^p(G,\omega)} \\
&\quad + \left|-\frac{1}{\pi}\int\int_{C\bar{D}} \frac{(S_n\circ y)(\psi(w))}{g[\psi(w)]w^{m+1}} y_{\bar{\varsigma}}[\psi(w)]\overline{\psi'(w)}d\sigma_w - b_m\right|
\end{aligned} \tag{18}$$

for every natural number n. Since $\|f - S_n\|_{A^p(G,\omega)} \to 0$, for $n\to\infty$, (17) and (18) show that $a_m(f,g) = b_m$ and the proof is complete.

As follows from this theorem, the generalized Faber coefficients of the function f are independent of the choice of canonical quasi-conformal reflections.

Proof of Theorem 3 Let P_n^* be the best approximation polynomial to $f \in A^2(E_R,\omega)$ in the norm $\|\cdot\|_{A^2(E_R,\omega)}$, *i.e.*,

$$\|f - P_n^*\|_{A^2(E_R,\omega)} = E_n(f, E_R, \omega),$$

and let $y(R,z)$ be a K_R-quasiconformal reflection across the boundary L_R. Since the function g is analytic and nonvanishing in CE, it belongs to the class $B_p(\overline{CE_R})$ and then according to Theorem 1 and Lemma 1 the equality

$$|f(z) - S_n(z,f,g)| = \left| \sum_{m=n+2}^{\infty} a_m(f,g) F'_m(z,g) \right|$$
$$= \frac{1}{\pi} \left| \int\int_{|w|>R} \frac{(f - P_n^*) \circ y(R,\psi(w)) \overline{\psi'(w)} y_{\bar{\varsigma}}[R,\psi(w)]}{g[\psi(w)]} \sum_{m=n+2}^{\infty} \frac{F'_m(z,g)}{w^{m+1}} d\sigma_w \right|$$

holds for every $z \in E$. Further applying Hölder's inequality and Lemma 2 we obtain

$$|f(z) - S_n(f,z,g)|^2 \le \frac{E_n^2(f, E_R, \omega)}{(1 - k_R^2)^2} \sum_{m=n+2}^{\infty} \frac{|F'_m(z,g)|^2}{mR^{2m}}.$$

Integrating both sides of last relation over E by virtue of Lemma 3 we get

$$\| f - S_n(\cdot,f,g) \|_{A^2(E)}^2 \le c_5 E_n^2(f, E_R, \omega) \left(\frac{r^2}{R^2} \right)^n.$$

From this

$$\| f - S_n(\cdot,f,g) \|_{A^2(E)} \le c \left(\frac{r}{R} \right)^n E_n(f, E_R, \omega),$$

where

$$c = \frac{\sqrt{\pi} M(g,r) r}{(1 - k_R^2) \sqrt{(R^2 - r^2)(r^2 - 1)}},$$

and our proof is completed.

Proof of Theorem 4 The proof go similarly to that of Theorem 3, using the Corollary 2, instead of Lemma 3.

Acknowledgement

The author is indebted to the referee for valuable suggestions.

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