Trigonometry Learning

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Abstract

Background: Trigonometry is an area of mathematics that students believe to be particularly difficult and abstract compared with the other subjects of mathematics. Trigonometry is often introduced early in year 8 with most textbooks traditionally starting with naming sides of right-angled triangles. Students need to see and understand why their learning of trigonometry matters.

Aims: In this study, particular types of errors, underlying misconceptions, and obstacles that occur in trigonometry lessons are described.

Sample: 140 tenth grade high-school students participated in the study. 6 tenth grade mathematics teachers were observed.

Method: A diagnostic test that consists of seven trigonometric questions was prepared and carried out. The students' responses to the test were analyzed and categorized. Observations notes were considered.

Results: The most common errors that the students made in questions were selected. Several problematic areas have been identified such as improper use of equation, order of operations, and value and place of sin, cosine, misused data, misinterpreted language, logically invalid inference, distorted definition, and technical mechanical errors. This paper gives some valuable suggestion (Possible treatment of students' error obstacles, and misconceptions) in trigonometric teaching for frontline teachers.

Conclusion: The study found students have errors, misconceptions, and obstacles in trigonometry lessons.

Keywords: instructional misconceptions, learning trigonometry, obstacles.

學習三角數學

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摘要

背景：比較其他數學的主題，三角學是學生認為特別困難和抽象的。學生通常在第八年級開始接觸三角學，傳統上多數課本以介紹直角三角形的邊作開始。學生需要瞭解他們為什麼要學三角學。

目的：這項研究觸及有關三角學習特殊錯誤的類型、其蘊含的誤解和在課堂上發生的障礙。

樣本： 140名第十年級高中學生和六位有關的數學教師　加了本研究。

方法：學生們參加了一次包括七個三角問題的診斷測試，其反應和觀察筆記被分類分析。

結果： 列出了學生答題時的通病，像誤用方程式、運算程序、正弦和餘弦的數值和位置、誤用數據、解錯題、錯誤邏輯推理、曲解定義、及技術運算錯誤等。本文給前線老師在教授三角學時可貴的建議［學生的錯誤障礙和誤解可能的治療］。

結論： 本研究找出了學生在學習三角課時所犯的錯謬、誤解、和障礙。

關鍵詞：授課誤解、學習三角、障礙

INTRODUCTION

Mathematics, particularly trigonometry is one of the school subjects that very few students like and succeed at, and which most students hate and struggle with. Trigonometry is an area of mathematics that students believe to be particularly difficult and abstract compared with the other subjects of mathematics.

Three generalizations were made because of their relationship to Piaget's description of formal operations that could be drawn from the study on misconceptions. These three generalizations are:

1. Many misconceptions are related to *a concept that produces* a mathematical object and symbol. For example: sine is a concept and symbol of trigonometric functions.
2. Many misconceptions are related to *process*: the ability to use operations. For example: as representing the result of calculation of $\sin 30^0$ and value of $\sin 30^0$.
3. Many misconceptions are related to *procept* that is, the ability to think of mathematical operations and object. Procept covers both concept and process. For example: sinx is both a function and a value. In addition to this, Gray and Tall (1994) asserted that "procedural thinking," that is, the ability to think of mathematical operations and object as procept, is critical to the successful learning of mathematics.

Many studies concerned with mathematics education explain that students have misconceptions and make errors, and these situations grow out of learning complexities (Lochead & Mestre 1988; Ryan&Williams, 2000). Of late, a few researchers have mentioned students' misconceptions, errors, and related to these, learning complexities about trigonometry (Delice 2002; Orhun 2002). Fi (2003) states that much of the literature on trigonometry has focused on trigonometric functions. Fi's study is related to the preservice secondary school mathematics teachers' knowledge of trigonometry: subject matter content knowledge, pedagogical content knowledge, and envisioned pedagogy. A few researchers studied more specific issues in trigonometry such as simplification of trigonometric expressions and metaphors (Delice, 2002; Presmeg 2006, 2007). Brown (2006) studied students' understanding of sine and cosine. She reached a framework, called trigonometric connection. The study indicates that many students had an incomplete or fragmented understanding of the three major ways to view sine and cosine: as coordinates of a point on the unit circle, as a horizontal and vertical distances that are graphical entailments of those coordinates, and as ratios of sides of a reference triangle… (p. 228). Orhun (2002) studied the difficulties faced by students in using trigonometry for solving problems in trigonometry. Orhun found that the students did not develop the concepts of trigonometry certainly and that they made some mistakes. The teacher-active method and memorizing methods provide students knowledge of trigonometry only for a brief moment of time, but not this knowledge is not retained by the students in the long run. Delice (2002identified five levels for measurement of students' knowledge about research theme and for defining students' skills. According to the results of the research, students partially answered the questions at the first and second levels and inadequately answered the questions at the other levels. The students have misconceptions and learning complexities, which is attributed to the fact that before learning the trigonometry concepts, the students learn some concepts, pre-learning concepts, incorrectly or defectively. These concepts are fundamental for learning the concepts of the trigonometry such as unit circle, factorization, and so on. Delice (2002) has a main assumption about the research that, generally speaking, errors are not random but results from misconceptions and that these misconceptions need to be identified in the study of trigonometry. Therefore, the students could not learn the procedure of solving the verbal problems confidently. Hogbin (1998) also reported similar findings. Additional findings by Delice (2002) indicated that Turkish students did much better with the algebraic, manipulative aspects of trigonometry and that English students did better with the application of trigonometry to practical situations in England. This article reports upon particular aspects of a study, the main aim of which was to compare the performance of students in the 16–18-age group from Turkey and England on trigonometry and then to compare the curriculum and assessment provision in each country to seek possible explanations for differences in performance. Weber (2005) investigated trigonometric functions in a study, which involved students of two colleges. One group was taught trigonometric functions traditionally and second group was taught according to Gray and Tall's (1994) notion of *procept,* current process object theories of learning. He found that second group that was taught by Gray and Tall's theories understood trigonometric functions better than the other group.

Guy Brousseau claims that the errors committed by students or the failure of students are not as simple as we used to consider in the past. The mistake not only results from ignorance, uncertainty, or chance as the empirical theory of knowledge used to claim. The mistake is the result of the previous knowledge that used to be interesting and successful, but now it has been proved wrong or simply uneducable. Mistake of these kinds are not irregular and unpredictable and these mistakes are due to obstacles. In the function of both the teacher and student, the mistake is a constituent part of the acquired knowledge. The present article distinguishes among the different mistakes committed by students, which result from obstacles and misconceptions.

However, many errors are committed due to the mechanical application of a rule in the trigonometry exercises. The researchers believe that some of the student's errors are related to the concept of "didactic contract". As Guy Brousseau (1984) says "in all the didactic situations a negotiation of a didactic contract is taking place, which defines, partly explicitly but mainly implicitly, what each partner has to do and for which, in a way or another he is held responsible towards the other. In another part he writes "Thus, in the didactic contract three elements are present: the pupil (the person who is taught), the teacher (the person who teaches), who are the partners, and the knowledge, as "material to be taught". The role played by the didactic contract is that of settling the interaction between the teacher and the pupils in connection with some knowledge. For example, the research of Bagni (1997) "Trigonometric Functions: Learning and Didactical Contract" gives evidence that in many trigonometric exercises the didactical contract forces the students to find always the solution to exercises that have no solution.

Description of the Research

The study focused on five objectives: What are the errors committed by students in trigonometry? What is a possible categorization of these errors and obstacles? What are the misconceptions and obstacles relating to learning trigonometric concepts? What are the possible treatments of students' errors, obstacles, and misconceptions? What are the student's answers that help us explore the students' thinking and reflection about learning?

Participants

In Turkey, the trigonometry is taught to students in mathematics lessons during the 10^{th} grade of High school, which is also called lyceum (i.e. age 14-17: High-school education covers the 4 years over the 14–17 age range). The students meet comprehensive trigonometry instruction at second semester of the10^{th} grade. Participants were chosen at random but proven not to be a representative sample. The sample was taken from different high schools, which have very different backgrounds. Teachers conducted a trigonometry diagnostic test to all 140 students.

Instruments

An interview was carried out with six 10^{th} grade mathematics teachers to learn the problems of teaching mathematics. The researcher also made a four week observation in the 10^{th} grade mix ability mathematics class (Table1). *The participants' past experiences about trigonometry:* In Turkey, the *trigonometry* concept is first taught to students in mathematics lessons in the 8^{th} grade of Elementary schools (i.e. c. age 9-10: Elementary Education covers the 8 years over the 6-14 age range). Brief explanations of right angle are given in the 8^{th} grade (ages 12-13) and a general introduction to *trigonometry* is made in the 10^{th} grade (ages 14-15). The formal mathematics courses, which go on for three years, start with secondary education, which is also called high school or Lycée. During the observation students and teachers investigated about trigonometry teaching and learning. The mathematics teachers in Turkish secondary schools usually prefer teaching with traditional techniques. In mathematics teacher training course, complex analysis (3credits), applied mathematics (3credits), history of mathematics (2credits) and calculus (6 credits) course includes trigonometry subject. But mathematics teacher trainees have a little practice of teaching trigonometry. Mathematics teachers tend to concentrate on solving the problems through algorithmic approaches, rather than concept learning. It is considered that practicing examples in this way is the best preparation for the university entrance examination (OSS). Additionally, the fact that the high school mathematics curriculum prescribes a lot of material to be covered is perceived as a real barrier to an emphasis on conceptual learning.

Table 1:

		Male		Female	
		2 teachers	1 teachers	2 teachers	1 teachers
Interview	Age	25-35	51	24-35	52
	Teaching experience	2 years 11 years	23 years	2 years 10 years	24 years
	Present teaching course	10^{th} grade (mix ability class)	10^{th} grade (mix ability class)	10^{th} grade (mix ability class)	10^{th} grade (mix ability class)
Observation	four weeks observation during trigonometry teaching (mix ability class)				

The interview extracts were analyzed and 20 trigonometry questions were established. The diagnostic test was applied to thirty-two 10^{th} grade students for piloting. After the pilot study, three university lecturers, 10^{th} grade mathematics teachers, reviewed items of the diagnostic test and researchers tailored it. The last form of diagnostic test included seven questions (Appendix 1).

Analysis and Scoring

The seven questions have been coded and analyzed according to a concept-evaluation scheme (Weiss & Yoes, 1991). All the answers to the questions were itemized and categorized from I to IV using the scoring criteria (Table 2). Also, diagnostic test paper of each student was labeled 1 to 140.

If the answer was coded as a *correct answer*, student gave all components of the validated response and correct answer (I). For example, for the right-angle triangle $(a/c)^2+(b/c)^2=(a^2+b^2)/c^2=c^2/c^2=1$.

If the answer was coded as a *partial understanding, the student g*ave at least one of the components of the validated response, but not all the components of the correct answer or just given concept process. This section was divided into two categories: Error in mechanical application of a rule. Error is related to concept of teachers' teaching, students' learning, and knowledge. Forexample:$x^2+y^2=sin^2x+cos^2x=1$ (process). If the students verified the identity for only a single case, it is also called partial understanding. e.g. $sin^2 30^0+cos^2 30^0=(1/2)^2+(\sqrt{3}/2)^2=4/4=1$

If the answer was coded as a *Misconception or obstacle, the student gave* illogical or incorrect information as an answer. Misconception covered mistakes and obstacles. It is the result of previous knowledge and obstacles.

For example: $1/(cos^2x)+1/(sin^2x)=1$ *and* $sin^2x.cos^2x=1$ (procept). Although it compromised concept and process, it did not give the result of this question. In the equation of sin^2x+cos^2x was used and the value of equation was not found as $sin^2x.cos^2x=1$. Justifying why $sin^2x.cos^2x=1$ involved reasoning about a process that could be used to produce the value of $sin^2x.cos^2x=1$. Similarly, justifying why "$sin^2x.cos^2x=1$" *was not the range of sinx, cosx,* sin^2x, cos^2x *involved understanding no matter what the input for this process.* If the answer was coded as *an unacceptable, the students gave Irrelevant or unclear responses, or not answered or irrelevant answers or repeat* information in the question as if it was an answer or blank. Thus, the coding schemes were developed and the respondents' ideas were coded. The frequencies were calculated. If the students mentioned these items, we calculated as a percentage. Three researchers performed coding, and intercoder reliability was found to be 89%. All represented findings included answers, which is given as italics.

Table 2:

	Criteria for Scoring
I-Correct answer	Included all components of the validated response, correct answer
II-Partial Understanding, Misconception or obstacle	*At least one of the components of the validated response, but not all the components, just concept or process or mechanical application of a rule, did not involve any justification *Included illogical or incorrect information or information different from the correct information
III-Unacceptable	Irrelevant or unclear responses or not answered or irrelevant answers, repeat information in the question as if it was an answer or Blank

The examples presented in this article are errors committed by students, obstacles and misconceptions that arose from high-school lessons in trigonometry.

FINDINGS

The present study is different from the other studies of trigonometry in terms of sampling, methodology, data analysis techniques, and findings. There are many errors, obstacles, and misconceptions in trigonometry, and these are given in this section.

Question 1: $\sin^2 x + \cos^2 x = 1$. Why? Please explain.

The list of students' writing of the question 1 related to the "$\sin^2 x + \cos^2 x = 1$" was coded and presented below according to the various levels of understanding. Students' justifications are given below:

I. *Correct answer* (77 responses out of 140)

$(a/c)^2+(b/c)^2= (a^2+b^2)/c^2=c^2/c^2=1$ (*If a, b, and c indicate the sides of a triangle*)$(a^2+b^2)/c^2=c^2/c^2=1$ (*If a, b, and c indicate the sides of a triangle*)

$\sin^2 x+\sin^2(90-x)=a^2/(b^2+a^2)= b^2/(b^2+a^2)=1/1=1$

II. **Partial Understanding* (49 responses out of 140)

$\sin^2 x= 1-\cos^2 x$ (process)

Already proved in a unit circle. (concept of unit circle)

$\sin^2 30^0+\cos^2 30^0= (1/2)^2+(\sqrt{3}/2)^2=4/4=1$ (process)

Thirteen students gave a mathematically valid explanation for why this equation was true. Ten out of thirteen students exemplified that this was true using a right-triangle model. These results were similar to the finding of Weber's study. Ten students demonstrated the identity for only a single case such as $\sin^2 30^0+\cos^2 30^0= (1/2)^2+(\sqrt{3}/2)^2=4/4=1$ (process).

$\sin^2 x + \cos^2 x=1$ formula can only be obtained from the Pythagoras equation (concept).

$x^2+y^2= \sin^2 x+ \cos^2 x=1$ (process)

******Misconception or obstacle* (12 responses out of 140)

$1/(\cos^2 x)+1/(\sin^2 x)=1$; $\sin^2 x. \cos^2 x=1$ (procept)

In the equation of $\sin^2 x + \cos^2 x$ was used and the value of equation was not found as $\sin^2 x. \cos^2 x=1$ *used $a^2-b^2= (a-b)(a+b)$ formula but did not get the result.* (process)

sinx and cosx are only defined in a unit circle. (concept)

There is an inverse relation between sinx and cosx.

$(\sin x+\cos x)^2 = \sin^2 x+ 2\cos x.\sin x+ \cos^2 x$

$(\sin x-\cos x)^2 = \sin^2 x- 2\cos x.\sin x+ \cos^2 x$

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$\sin^2 x+\cos^2 x=(\sin x-\cos x)^2+(\sin x+\cos x)^2$ (concept)

In the equation of $\sin^2 x + \cos^2 x$ was obtained, but the value of equation was not 1.

III. *Unacceptable* (2 responses out of 140**):** two students gave the answer as a blank.

According to results, 55% of the students showed an understanding of this question: 35% of the students had a partial understanding of the question; nearly 8% of the students showed misconception statements, which were identified through analysis of the question, and five misconception statements were identified through analysis of the question $\sin^2 x + \cos^2 x = 1$ and unit circle defined as sin2x. cos2x. Therefore, the result is 1. Most students only simply memorized the formula and they calculated it using the unit circle. This can be attributed to the fact that course textbooks and teachers almost always introduced this subject using the same method as was shown by the students. The students retained the aspects learnt by them in secondary school when they moved on to high school. The students knew only the $\sin^2 x + \cos^2 x=1$ equation, but could not explain it. The other reason for this was the secondary-school

mathematics textbooks in Turkey. The following statements clearly identify this misconception:
"*If a numeric value is given, then I could have calculated*", "*if a proving question* is *asked, and I could have proved it"* (Student 13: S13)

Question 2: tanx = $\frac{1}{\cot x}$ or tanx . cotx = 1. Please explain why?

"I never thought that $sin^2x + cos^2 x$ is equal one?" and *"we use a^2-b^2= (a-b)(a+b) formula in calculations"*(Student 20:S20)

Common errors, obstacles, and misconceptions that students made with probe "**tanx.cotx=1**" equation is highlighted. The justifications given by students are shown below:

I. *Correct answer* (130 responses out of 140)

tanx=(1/(cotx))=(1/(cosx/sinx))= sinx/cosx (6 responses out of 140)

tanx a/b, cotx=b/a; a/b=(1/(b/a)) then (a/b)=(a/b)=1 (100 responses out of 140)

sinx ≠0 and cosx≠0 (11 responses out of 140)

tanx.cotx=1 is always give 1. (8 responses out of 140)

It is the opposite of each other (5 responses out of 140)

Surprisingly, 130 students answered this question correctly. But they only memorized the definition without any understanding.

II. **Misconception or obstacl*e (9 responses out of 140)

It gives 1=1. (5 responses out of 140) (There was no explanation)

tanx and cotx in relation to each other were complete at 360^0 (4 responses out of 140

III. *Unacceptabl**e*** **(**1 response out of 140)

Their answers showed that the students had found the correct answer because most of them knew the formula of tanx and cotx. However, they wrote only simplified answers, which showed only partial understanding of the question. Examples of the answers they gave were, "because it's opposite/backwards/the wrong or other way round" as well as, "is always multiplied by 1". From this, it can be seen that students simply memorized the formula found in the textbooks, e.g. 7% of the students showed a misconception for the question:

"1 = 1" (S 139, S 110, S5, S47, S72)

"tanx . cotx" always completes each other to 180^o. (S139 S8, S81, S72)

These answers were classified as misconceptions. A statement in such a book potentially encouraged this kind of misconception, as stated below:

"If a radius is equal to one, this is a circle equation. This is called unit circle" (Barış, 2000). The definition shown in the books encourage students to develop misconceptions. The list of students' writing related to the "**tanx.cotx =1**"was coded and presented below according to the various levels of understanding of the students who showed understanding of this question.

Question 3: tan 90 is undefined, Please explain why?

Common errors, obstacles, and misconceptions that students make with Probe "**tan 90 = undefined**" equation is highlighted. Students give justifications below:

I. *Correct answer* (132 responses out of 140)

tan 90 = sin90/cos90 = 1/0, undefined

Number / 0 = ∞, then it is undefined

II. **Partial Understanding* (4 responses out of 140)

If $x=90^{0,}$ then the tangent curve did not intersect x=1 and x=-1 lines

*Misconception or obstacle (3 responses out of 140)

1.tan90= hypotenuse/adjacent line

III. *Unacceptable* (1 response out of 140), no response

The list of students' writing related to the "tan90=undefined" was coded and presented below according to the various levels of understanding. Most of the students understood the question. Only 3 of the students showed misconception of this question. The reason for the correct answers might be that the students memorized the expressions they learned from their course books. The students just memorized everything from the textbook, regardless of what they were taught in class.

Question 4: What is $(\sin^2 x)^2 - (\cos^2 x)^2 =$? Please simplify.

The list of students' writing related to the "$(\sin^2 x)^2 - (\cos^2 x)^2 =$"was coded and presented below to the various levels of understanding. Common errors, obstacles and misconceptions that students make with probe "$(\sin^2 x)^2 - (\cos^2 x)^2 =$?"equation is highlighted. The justifications given by students are given below:

I. *Correct answer* (54 responses out of 140)

No result to the differences of two squares; $\sin^2 x - \cos^2 x$

$1-(\cos 2x - \sin^2 x) = -\cos 2x$

II. **Partial Understanding* (11 responses out of 140)

No result to the differences of two squares

It is only considered the square of the parenthesis; they did not consider the inside of parenthesis

**Misconception or obstacle* (75 responses out of 140)

Above equation is equal to $\sin^2 x=1-\cos^2 x$ (students memorized this equation) (50 responses out of 140)

*The equation is equal to $1-2\cos^2 x$. (*Students were memorizing this equation*)* (25 responses out of 140)

Approximately 39% of the students displayed an understanding of the question, and 85 of the students showed partial understanding. These answers showed that the students had realized the difference between two squares. Most of the students showed misunderstanding of the question. It can be seen that students simply memorized information and were unable to transfer this knowledge to another situation.

Question 5: tanA=3/4 then what is tan2A=? Please explain.

Common errors, obstacles, and misconceptions that students make with Probe "***tanA=3/4*** then ***tan2A***" equation is highlighted. Students gave justifications as shown below:

I. *Correct answer* (34 responses out of 140)

Let us consider 3,4,5 right triangle.

1.tanA=3/4,then tan2A=sin2A/cos2A=2sinA.cosA/(cos^2A-sin^2A)=24/7

2.tan2A=sin2A/cos2A=24/7 (16 responses out of 140)

II.**Partial Understanding* (24 responses, 18%)

The formula tan (A+B) could not be converted to tan2A.

Only a formula can be used

Error was committed during working out of the answer

**Misconception or obstacle* (72 responses out of 140)

'I did not know which side of the triangle is called the adjacent edge or opposite edge'. 90^0 (13 responses, out of 140)

a^2=9+16-8a+a^2thena=25/8; tan2A=3/(4-a)= 3/(4-25/8)= 24/7 (17 responses out of 140)

tan2A = $(3/2)^2$=9/16 (14 responses out of 140)

sinA/cosA = 3/4 (15 responses, out of 140)

tan2A=1+tan^2A= 1-tanA.cotA=1+(3/4). (4/5)=9/5 (13 responses out of 140)

III. *Unacceptable* (10 responses out of 140), no responses

The list of students' answers related to the "***tanA=3/4* than *tan2A***" question was coded and is presented below according to the various levels of understanding. Students who learnt the formula were unable to apply the formula in different situations. 51% of the students showed misunderstanding of the question. They were unable to remember the formula as they only memorized it.

Question 6: cos(-θ) = cosθ . Please explain why?

The list of students' writing related to the "cos(-θ) = cosθ "was coded and presented below according to the various levels of understanding. Common errors, obstacles, and misconceptions that students make with Probe

"**cos(-θ)=cosθ** "equation is highlighted. Students give the justifications below:

I. *Correct answer* (71 responses out of 140**)**

The function of cosx was positive in region I and in region IV of the unit circle (they drew a unit circle)

It was a double function

II. **Partial Understanding* (27 responses out of 140)

cos(-37)=cos37 then 0.8=0.8. and cos (-45)=cos45

cosx will be positive in region IV (10 responses out of 140)

* *Misconception or obstacle* (36 responses out of 140)

-θ = *-180* (4 responses out of 140)

n is in the area II (5 responses out of 140)

*The value of "***cos (-θ) = cosθ** *" was same everywhere* (10 responses out of 140)

It was the same for every value (8 responses out of 140)

The symbols do not change (9 responses out of 140)

III. *Unacceptable* **(**6 responses out of 140)

The students stated, *"It is easier to work out the equation if they are given values to work with, and if the angle would be equal in the second area*". In addition, they said that "*every value would be the same,"* and that "*the symbol would not change*". The memorized formula was either forgotten, or remembered incorrectly. Some other students wrote that they did not have sufficient knowledge to answer the question.

Question 7: Why are tangent and cotangent functions positive in Region III of the unit circle? Please explain?

The list of students' writing related to the "why are the tan and cos functions positive?" was coded and presented below according to the various levels of understanding. Common errors, obstacles, and misconceptions that students make with probe "**Why are the functions of tanx and cotx positive in region III of the unit circle**?" Students gave the justifications as shown below:

I.*Correct answe**r*** (118 responses, 84%)

tanx = sinx/cosx= -/- = +

cotx= cosx/sinx=-/- = +

\- + + +

\- - + - (108 responses out of 140)

II. **Partial Understanding* (5 responses, 4%) *x and y have negative values* (5 responses out of 140)

**Misconception or obstacle* (17 responses, 12%)

tanx = -1 / (-1) = 1 (7 responses out of 140)

cotx = -1 / (-1) = 1 (7 responses out of 140)

By definition is positive (3 responses out of 140). They understood the values for tanx and cosx to be 1, and because of its definition, they thought it would be positive.

DISCUSSION AND CONCLUSION

Identifying and helping students overcome obstacles and misconceptions includes 5 subsections that give an answer for each research question.

What are the sources of errors committed by students?

The results of this study showed that students have some misconceptions and obstacles about trigonometry. One of the two obstacles to effective learning was that trigonometry and other concepts related to it were abstract and non-intuitive. Lochead and Mestre (1988) described an effective inductive technique for these purposes. The technique may be induced conflict by drawing out the contradictions in students' misconceptions. In the process of resolving the conflict, a process that takes time, students reconstruct the concept (Ubuz, 1999, 2001). The students had problems with prior and new knowledge about concept, process, and procept in learning trigonometry. The reasons of errors, which students made in trigonometry lesson, were mal-rule teaching or teaching concepts. This may be especially important at the introductory level. It is caused from their habits, as well as the development of inaccurate constructions, on the part of the learner. It may also be useful for the teacher, when recognizing a specific error, to point it out to the students for, as Borasi (1994, page 166) observed, "although teachers and researchers have long recognized the value of analyzing student errors for diagnosis and remediation, students have not been encouraged to take advantage of errors as learning opportunities in mathematics instruction." The teacher has an important role to play in overcoming it. Teacher's roles are to observe the students, and if they are making mistakes and errors; s/he could discuss and correct them.

What is a possible categorization of these errors?

Students make several different reasoning errors in trigonometry. Some of these errors were based on underlying obstacles and misconceptions, while others, although repeatedly observed, were of a partial understanding or misconception. Students persist in making both types of errors.

All the answers of the students to the questions were itemized from I to IV using the scoring criteria in Table 1. The classification of general errors was based on model of Movshovitz-Hadar *et al*. (1987). The theoretical assumption was that "the most of students' error in high-school mathematics are not accidental and are derived by a quasi-logical process that somehow makes sense to the student" (p.3–4). This model includes five descriptive categories of errors and misconceptions, which were identified in the present research, except one category, unverified solution. Excluding general errors, the study considered errors that resulted from misconceptions of procedures and concept linked to trigonometry.

The categories *of trigonometry misconceptions were:* misused data, misinterpreted language, logically invalid inference, distorted definition, and technical mechanical errors.

Misused data: e.g. *tanx . cotx multiplication always gave 1=1*

Misinterpreted language: These misconceptions were related to *a concept that produced* a mathematical object and symbol (e.g. *'I did not know which side of the triangle is called the adjacent edge or opposite edge'. 90^0*; *A value of "***cos (-θ) = cosθ "** *is same everywhere),*

Logically invalid inference: These misconceptions were related to *procept* (e.g. *sinx and cosx are only defined in a unit circle;* $sin^2x + cos^2 x= (sinx-cosx)^2 +(sinx+cosx)^2$; t*anx and cotx in relation to each other are complete at* 360^0 and *it is the same for every value),*

Distorted definition: the misconceptions were related to *procept,* that is, the ability to think of mathematical operations and object. Procept covered both concept and process. (e.g. *$sin^2x.\ cos^2x=1$, for every x; 1.tan90= hypotenuse/adjacent line*),

Technical mechanical errors: These misconceptions were related to *process*: the ability to use operations (e.g. *The equation is equal $1-2cos^2x$; $a^2=9+16-8a+a^2$ then a=25/8; tan2A=3/(4-a)= 3/(4-25/8)= 24/7)*.

To sum up, we sometimes speculate on the underlying causes of the errors, but do not see them as conceptual in nature. One of the following errors such as "usage of simple formulae," "calculation of trigonometric values of angles," and "to populate the values into a formula" was determined to be correct by one or more students. Few students made mistakes choosing the correct ratio to use, but the manipulation caused some problems. For example, some students seemed not to be sure whether s/he should use the trigonometric value of the angle. In addition to the above-mentioned problems, few students also had problem with notation.

What are the obstacles and misconceptions relating to concepts of learning trigonometrically?

We offered our views as to the possible underlying obstacles and misconceptions, that is, we gave a general rule or idea, which, if believed by a student, would result in that type of error. These errors were considered to have a rational basis, and we comment on how they might come about. Another mistake made by students was when they found the result of a division when zero divides a number. The students gave up their errors, obstacles, and misconceptions, which can have such a harmful effect on learning, only with great reluctance. Not only do students bring their experiences, obstacles, and misconceptions to class, students suggested that repeating a lesson or making it clearer will not help those students who base their reasoning on strongly held misconceptions. The other obstacle is the exam of university entry. The university entrance examination should be more comprehensive, and more importance should be given to questions on trigonometry. Students should be well motivated while solving problems using four operations, formula and parenthesis about trigonometry. This set of results gave an indication that students are prone to common errors even when teachers have adopted different teaching strategies for teaching trigonometry.

What are the possible treatments of students' errors, obstacles, and misconceptions?

Students do not come to the classroom as "blank box". Instead, they come with their own ideas and theories constructed from their **everyday experiences**, and they use these theories. Another suggestion of treatment is using **resources, materials, diagrams, and equipments**. Explaining to each other helps improve the students' understanding, and enhances confidence in their own mathematical ability. This would also work while manipulating the formulae to try to `solve' a triangle combined with the use of the algorithmic diagrams and graphs. Presmeg (2006) emphasizes that connecting old knowledge with new and allowing ample time and moving into complexity slowly, connecting visual and no-visual registers like numerical, algebraic, and graphical signs are important for students, providing memorable summaries in diagram form, which have the potential of becoming prototypical images of trigonometric objects for the students, because these inscriptions are sign vehicles for these objects in trigonometry teaching.

In a trigonometry example, simple material needs to be used in order to show its spiral function. Before introducing a new topic, teachers need to know if students have enough knowledge. If the students do not have enough knowledge, or have any misconceptions, the teachers need to try and get rid of it. For example, before teaching trigonometry, the students need to give some examples of Pythagoras theorem in different triangles. Meaningful learning needs to be obtained. The meaningful learning aids that might be used include Vee Heuristics, concept map, mind map, etc. To sum up, the present study concludes with a discussion of techniques to help students overcome their errors, mistakes, and misconceptions in trigonometry. These errors, obstacles, or misconceptions can be got rid of with plenty of practice. Errors, obstacles, and misconceptions made by students working at the board can be picked up and corrected by the other students. When students feel comfortable in class, we can act with them, without the error costing us very much. But this view only indicates that getting rid of errors and misconceptions is an act of education that integrates thinking, feeling, and acting (DES, 1981). However, since misconceptions are within the domain of conceptual behavior and since they are based on a belief about a certain mathematical situation, it makes it easier to engage them in classroom activities rather than simply try to correct them (Ryan & Williams, 2000).

Another area of weakness was in considering long-term action to reduce or avoid misconceptions. Both the students and teachers recognized that the work on misconceptions had many benefits in addressing areas such as planning, classroom observation, and evaluation. In doing this, it highlighted the inter-relationships between areas and thus avoided the fragmented and itemized approach that the initial training regulations could be seen to encourage. The intention is to retain and refine the student's diary and assignment for use in future years.

What are the student's answers that help us explore the students' thinking and reflection about learning.

Instead of that there is a descriptive presentation of student's answers without any kind of interpretation could be helping us to their thinking and reflecting about their learning. Students reflect their thinking into their writing. If we carefully analyze their writing, it is easy to understand them. Thus, teacher and student could understand each other.

PROSPECTS FOR FURTHER WORK

As a result of the study, to determine the misconceptions and obstacles about trigonometry, the researchers must investigate the students' cognitive processes. The determination of the students' misconceptions, obstacles, and errors also involves a qualitative study, so it must be a deep study. For an in depth understanding of the students' problems in trigonometry, the interviews can be done with the students.

APPENDIX A: DIAGNOSTIC TEST

Section 1

1. $\sin^2 x + \cos^2 x = 1$. Why? Please explain**.**

2. $\tan x = \frac{1}{\cot x}$ or **$\tan x \cdot \cot x = 1$.** Please explain.

3. Tan 90 is undefined. Please explain.

4. $(\sin^2 x)^2 - (\cos^2 x)^2$? Please simplify.

5. TanA=3/4 then tan2A=?

6. $\cos(-\theta) = \cos\theta$. Please explain why?

7. Why are tangent and cotangents functions positive in Region III of the unit circle? Please explain.

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