

REGULAR AND π – INVERSE MONOIDS UNDER
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Received April 25, 2011

Abstract

In this paper we give a partial answer to the problem which is about the regularity of Schützenberger product in semigroups asked by Gallagher in his thesis [4, Problem 6.1.6]. Furthermore, we determine necessary and sufficient conditions for the new version of the Schützenberger product of monoids which was firstly defined in [1] to be regular and strongly π -inverse.

AMS Subject Classification: 20E22, 20M15, 20M18.**Key words:** Schützenberger product, π -inverse monoid, regular monoid.

*This work was partially supported by TÜBİTAK with Programme 2221.

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1 Introduction and Preliminaries

In [4, Problem 6.1.6], Gallagher asked whether there exists a classification for arbitrary semigroups A and B for which the Schützenberger product $A \diamond B$ is regular. In fact, before asking this problem, the question of the regularity of the wreath product of monoids was explained by Skornjakov ([12]). After that, in [8], it has been investigated regular property of semidirect products of monoids. After these works, in [11], it has been investigated inverse and orthodox properties of semidirect and wreath products of monoids. It further has been shown in [13] that the restriction which the author imposed on one factor of the semidirect product in [11] was not necessary. To convince the above problem, as the first main result of this paper, we purpose to give a partial answer by defining necessary and sufficient conditions of the Schützenberger product $A \diamond B$ to be regular where both A and B are any monoids.

It is well known that wreath products are one of the most important constructions both for groups and semigroups. Indeed, they are one of the basic tools in the theory of permutation groups and also in the famous Krohn-Rhodes decomposition theorem for finite semigroups ([7]). Besides it has played a central role in many algebraic and geometric properties (see, for example, in [2, 3]). It is therefore of interest to improve this product to some other new constructions. In this direction one step has been taken by Ateş and Çevik ([1]). In [1], the authors defined a new monoid construction under wreath product (which is called a new version of the Schützenberger product of monoids) and then studied some algebraic properties (e.g., periodicity and locally finite) of this product. Therefore as a next step of [1], in this paper, as the other results, we will give regularity and strongly π -inverse property of this new product. This s

A generating and defining relation sets for the Schützenberger product of arbitrary monoids have been defined in a joint paper in [5]. Moreover, in [4], Gallagher defined the finitely generatability and finitely presentability of this product and then he left an open problem explained in the above first paragraph.

Let A and B be any monoids with associated presentations $\varphi_A = [X; R]$ and $\varphi_B = [Y; S]$, respectively. Each paragraph at the rest of this section,

we will recall definitions of some products which will be needed for the main results of this paper.

Let $M = A \rtimes_{\theta} B$ be the corresponding semidirect products of these two monoids, where θ is a monoid homomorphism from B to $End(A)$ such that, for every $a \in A, b_1, b_2 \in B, (a)\theta_{b_1 b_2} = ((a)\theta_{b_2})\theta_{b_1}$. We recall that the elements of M can be regarded as ordered pairs (a, b) , where $a \in A, b \in B$ with the multiplication given by $(a_1, b_1)(a_2, b_2) = (a_1(a_2)\theta_{b_1}, b_1 b_2)$, and the monoids A and B are identified with the submonoids of M having elements $(a, 1_B)$ and $(1_A, b)$. For every $x \in X$ and $y \in Y$, choose a word, denoted by $(x)\theta_y$, on X such that $[(x)\theta_y] = [x]\theta_{[y]}$ as an element of K . To establish notation, let us denote the relation $yx = (x)\theta_y y$ on $X \cup Y$ by T_{yx} and write T for the set of relations T_{yx} . Then, for any choice of the words $(x)\theta_y$, $\varphi_M = [X, Y; R, S, T]$ is a standard monoid presentation for the semidirect product M .

The cartesian product of B copies of the monoid A is denoted by $A^{\times B}$, while the corresponding direct product is denoted by $A^{\oplus B}$. One may think of $A^{\times B}$ as the set of all such functions from B to A , and $A^{\oplus B}$ as the set all such functions f having finite support, that is to say, having the property that $(x)f = 1_A$ for all but finitely many x in B . The unrestricted and restricted wreath products of the monoid A by the monoid B , are the sets $A^{\times B} \times B$ and $A^{\oplus B} \times B$, respectively, with the multiplication defined by $(f, b)(g, b') = (f {}^b g, bb')$, where ${}^b g : B \rightarrow A$ is defined by

$$(x) {}^b g = (xb)g, \quad (x \in B) \tag{1}$$

such that $(xb)g$ has finite support. It is well known that both these wreath products are monoids with the identity $(\bar{1}, 1_B)$, where $x\bar{1} = 1_A$ for all $x \in B$. (For more details on the definition and applications of restricted (unrestricted) wreath products, we can refer, for instance, [5, 7, 9, 10]). We should note that, for having finite support, B must be finite or a group.

Now for a subset P of $A \times B$ and $a \in A, b \in B$, we let define

$$Pb = \{(c, db) ; (c, d) \in P\} \quad \text{and} \quad aP = \{(ac, d) ; (c, d) \in P\}.$$

Then the Schützenberger product of A and B , denoted by $A \diamond B$, is the set $A \times \wp(A \times B) \times B$ (where $\wp(\cdot)$ denotes the power set) with the multiplication

$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1a_2, P_1b_2 \cup a_1P_2, b_1b_2)$. Clearly $A \diamond B$ is a monoid ([5]) with the identity $(1_A, \emptyset, 1_B)$.

We recall that a monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that $aba = a$ and $bab = b$ (or, equivalently, for the set of inverses of a in M , that is, $a^{-1} = \{b \in B : aba = a \text{ and } bab = b\}$, M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset). Also, for a semigroup S , we call S is an *inverse semigroup* if every element has exactly one inverse. The well known examples of inverse semigroups are groups and semilattices. In addition, let $E(S)$ and $RegS$ be the set of *idempotents* and *regular* elements of a semigroup S , respectively. In here, S is called *π -regular* if, for every $s \in S$, there is an $m \in \mathbb{N}$ such that $s^m \in RegS$. Moreover, if S is π -regular and the set $E(S)$ is a commutative subsemigroup of S , then S is called *strongly π -inverse semigroup*. We note that $RegS$ is an inverse subsemigroup of

2 Main Theorems

The following first theorem aims to give necessary and sufficient conditions for $A \diamond B$ to be regular while both A and B are arbitrary monoids.

Theorem 2.1 *Let A and B be any monoids. The product $A \diamond B$ is regular if and only if*

- (i) A and B are regular,
- (ii) for every $(a, P, b) \in A \diamond B$, either

$$P = aP_1b = \bigcup_{(a_1, b_1) \in P_1} \{(aa_1, b_1b)\} \text{ or } P = caP_1bd = \bigcup_{(a_1, b_1) \in P_1} \{(caa_1, b_1bd)\},$$

where $P_1 \subseteq A \times B$ and $c \in a^{-1}$, $d \in b^{-1}$.

By (1) and the definition of Schützenberger product, we can define a new version of the Schützenberger product as follows. (We note that the definition and some other properties of this product have been first given in [1]).

Let A and B be monoids. We recall that $A^{\oplus B}$ is the set of all functions f having finite support. For $P \subseteq A^{\oplus B} \times B$ and $b \in B$, we define the set

$$Pb = \{(f, db); (f, d) \in P\}.$$

The new version of the Schützenberger product of A by B , denoted by $A \diamond_v B$, is the set $A^{\oplus B} \times \wp(A^{\oplus B} \times B) \times B$ with the multiplication

$$(f, P_1, b_1)(g, P_2, b_2) = (f \cdot {}^{b_1}g, P_1 b_2 \cup P_2, b_1 b_2). \quad (2)$$

One can easily show that $A \diamond_v B$ is a monoid with the identity $(\bar{1}, \emptyset, 1_B)$, where ${}^{b_1}g$ is defined as in (1). We should also note that, for having finite support, B must be finite or a group.

Thus other main results of this paper are the following.

Theorem 2.2 *Let A be an arbitrary monoid and B be a finite monoid or be a group. Then $A \diamond_v B$ is regular if and only if*

- (i) A and B are regular,
- (ii) for every $x \in B$ and $f \in A^{\oplus B}$, there exist $e \in B$ such that $e^2 = e$ with $(x)f \in A(xe)f$,
- (iii) for every $(f, P, b) \in A \diamond_v B$, either

$$P = P_1 b = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 b)\} \quad \text{or} \quad P = P_1 b d = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 b d)\},$$

where $P_1 \subseteq A^{\oplus B} \times B$ and $d \in b^{-1}$.

Theorem 2.3 *Let A and B be two monoids. Then $A \diamond_v B$ is a strongly π -inverse monoid if and only if*

- (i) both A and B are strongly π -inverse monoids,
- (ii) for every $h \in E(A^{\oplus B})$ and $e \in E(B)$, there is ${}^e h = h$,
- (iii) for $f \in A^{\oplus B}$ and $e \in E(B)$, if $f {}^e f = f$ then ${}^e f = f$,

- (iv) for every $f \in \text{Reg}(A^{\oplus B})$ and $e \in E(B)$, $ef = f$,
- (v) for every $f \in A^{\oplus B}$ and $b \in B$, there exists $m \in \mathbb{N}$ such that $b^m \in \text{Reg}B$ and $f^{(m)} \in \text{Reg}(A^{\oplus B})$, where $f^{(m)} = f \cdot b f \cdot b^2 f \cdots b^{m-1} f$,
- (vi) for every $(f, P, b) \in A \diamond_v B$, either

$$P = P_1 b = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 b)\} \text{ or}$$

$$P = P_1 b^2 = \bigcup_{(f_1, b_1) \in P_1} \{(f_1, b_1 b^2)\} \text{ or}$$

$$\dots \text{ or } P = P_1 b^{m-1} \text{ or } P = P_1 b^m,$$

where $P_1 \subseteq A^{\oplus B} \times B$.

3 Proofs

Proof of Theorem 2.1: Let us suppose that $A \diamond B$ is regular. Thus, for $(a, \emptyset, b) \in A \diamond B$, there exists (c, P, d) such that

$$(a, \emptyset, b) = (a, \emptyset, b)(c, P, d)(a, \emptyset, b) = (aca, aPb, bdb),$$

$$(c, P, d) = (c, P, d)(a, \emptyset, b)(c, P, d) = (cac, Pbd \cup caP, dbd).$$

Therefore we have $a = aca$, $c = cac$, $b = bdb$ and $d = dbd$. This implies that (i) must hold.

By the assumption on the regularity of $A \diamond B$ and for an element (a, P, b) in the product $A \diamond B$, we have $(c, P_2, d) \in A \diamond B$ such that

$$(a, P, b) = (a, P, b)(c, P_2, d)(a, P, b) \text{ and } (c, P_2, d) = (c, P_2, d)(a, P, b)(c, P_2, d).$$

Hence this gives us $a = aca$, $c = cac$, $b = bdb$, $d = dbd$, $P = Pdb \cup aP_2b \cup acP$ and $P_2 = P_2bd \cup cP_2d \cup caP_2$. To show the second condition in theorem, let us suppose that $P \neq aP_1b$, for some $P_1 \subseteq A \times B$. Then there exists $(a_2, b_2) \in P$ such that $a_2 \neq aa'_2$ and $b_2 \neq b'_2b$ where $a'_2 \in A$ and $b'_2 \in B$. Thus P cannot be equal to $Pdb \cup aP_2b \cup acP$, for all $P_2 \subseteq A \times B$. This

gives a contradiction with the regularity of $A \diamond B$. In fact, if someone takes $P = aP_1b$, then the equalities

$$\begin{aligned} Pdb \cup aP_2b \cup acP &= aP_1bdb \cup aP_2b \cup acaP_1b \\ &= aP_1b \cup aP_2b \cup aP_1b \\ &= aP_1b \text{ by choosing } P_2 = caP_1bd \\ &= P \end{aligned}$$

and

$$\begin{aligned} P_2bd \cup cPd \cup caP_2 &= P_2bd \cup caP_1bd \cup caP_2 \\ &= caP_1bdb \cup caP_1bd \cup cacaP_1bd \\ &\text{by choosing } P_2 = caP_1bd \\ &= caP_1bd \cup caP_1bd \cup caP_1bd = caP_1bd = P_2 \end{aligned}$$

hold. We note that, by applying the above similar discussions for the case $P = caP_1bd$ in this theorem, where $P_1 \subseteq A \times B$ and $c \in a^{-1}$, it is easily seen that condition (ii) must hold.

For the converse part of the proof, let $(a, P, b) \in A \diamond B$. Thus we definitely have $c \in A$ and $d \in B$ such that $c \in a^{-1}$ and $d \in b^{-1}$. Now let us consider the union of sets $Pdb \cup aP_2b \cup acP$ and $P_2bd \cup cPd \cup caP_2$. At this stage, by $P = aP_1b$, if we choose $P_2 = caP_1bd \subseteq A \times B$, then we get

$$Pdb \cup aP_2b \cup acP = aP_1b = P \quad \text{and} \quad P_2bd \cup cPd \cup caP_2 = caP_1bd = P_2.$$

As a result of this, for every $(a, P, b) \in A \diamond B$, there exists $(c, P_2, d) \in A \diamond B$ such that

$$\begin{aligned} (a, P, b)(c, P_2, d)(a, P, b) &= (aca, Pdb \cup aP_2b \cup acP, bdb) = (a, P, b), \\ (c, P_2, d)(a, P, b)(c, P_2, d) &= (cac, P_2bd \cup cPd \cup caP_2, dbd) = (c, P_2, d). \end{aligned}$$

In addition, by applying the above similar arguments for the case $P = caP_1bd$, where $P_1 \subseteq A \times B$ and $c \in a^{-1}$, the proof of the regularity of $A \diamond B$ is completed.

Hence the result. □

Furthermore it can be given the following result as a consequence of the Schützenberger product $A \diamond B$ of two monoids A and B to be regular.

Theorem 3.1 *The Schützenberger product $A \diamond B$ of two monoids A and B is regular if and only if both factors A and B are groups.*

Proof Let A and B be monoids and suppose that $A \diamond B$ is regular. Then, for each $a \in A$ and $b \in B$, the element $(a, \{(1_A, 1_B)\}, b)$ of $A \diamond B$ is regular, that is, there exists an element $(x, P, y) \in A \diamond B$ such that

$$(a, \{(1_A, 1_B)\}, b)(x, P, y)(a, \{(1_A, 1_B)\}, b) = (a, \{(1_A, 1_B)\}, b)$$

Hence,

$$\{(1_A, 1_B)\} = ax\{(1_A, 1_B)\} \cup aPb \cup \{(1_A, 1_B)\}yb,$$

and, in particular,

$$(ax, 1_B) = \{(1_A, 1_B)\} \quad \text{and} \quad (1_A, yb) = \{(1_A, 1_B)\}.$$

Altogether, the monoids A and B satisfy

$$\forall a \exists x : ax = 1_A \quad \text{and} \quad \forall b \exists y : yb = 1_B,$$

both of above statements imply that the monoids in assumptions are groups. The converse part, i.e. the regularity of two groups under Schützenberger product, is well known. \square

We note that, by some minor additional effort, one can show that the Schützenberger product $A \diamond B$ of two *semigroups* is regular if and only if A is a *right group* and B is a *left group*.

Proof of Theorem 2.2: Let us suppose that $A \diamond_v B$ is regular. Thus, for $(f, \{(1_A, 1_B)\}, b) \in A \diamond_v B$, there exists $(g, P, d) \in A \diamond_v B$ such that

$$\begin{aligned} (f, \{(1_A, 1_B)\}, b) &= (f, \{(1_A, 1_B)\}, b)(g, P, d)(f, \{(1_A, 1_B)\}, b), \\ (g, P, d) &= (g, P, d)(f, \{(1_A, 1_B)\}, b)(g, P, d). \end{aligned}$$

We then have $b = bdb$ and $d = dbd$. If we choose $b = 1$ then we get $d = 1$, and so $bd = 1$. Therefore we have $f = fgf$ and $g = gfg$. This implies that both B and $A^{\oplus B}$ are regular. Since $A^{\oplus B}$ denotes the direct product of B

copies of A , it is easy to see that if $A^{\oplus B}$ is regular, then A is regular. This gives condition (i).

By the assumption, for every $(f, P, b) \in A \diamond_v B$, we have $(g, P_2, d) \in A \diamond_v B$ such that

$$\begin{aligned} (f, P, b) &= (f, P, b)(g, P_2, d)(f, P, b) = (f {}^b g {}^{bd} f, Pdb \cup P_2b \cup P, bdb), \\ (g, P_2, d) &= (g, P_2, d)(f, P, b)(g, P_2, d) = (g {}^d f {}^{db} g, P_2bd \cup Pd \cup P_2, dbd). \end{aligned}$$

Hence, by equating the components, we get $f = f {}^b g {}^{bd} f$, $g = g {}^d f {}^{db} g$, $b = bdb$, $d = dbd$, $P = Pdb \cup P_2b \cup P$ and $P_2 = P_2bd \cup Pd \cup P_2$. These show that, for every $x \in B$,

$$(x)f = (x)f (x) {}^b g (x) {}^{bd} f = (x)f (xb)g (xbd)f \in A(xbd)f.$$

If we take $e = bd$, then condition (ii) becomes true. In addition, by using the facts $b = bdb$, $d = dbd$, $P = Pdb \cup P_2b \cup P$ and $P_2 = P_2bd \cup Pd \cup P_2$, for every $(f, P, b) \in A \diamond_v B$, and by applying similar arguments given in the proof of Theorem 2.1, we get

$$\text{either } P = P_1b \text{ or } P = P_1bd,$$

where $P_1 \subseteq A^{\oplus B} \times B$ and $d \in b^{-1}$. Therefore condition (iii) must hold.

Conversely, let us suppose that the monoids A and B satisfy conditions (i), (ii) and (iii). For $x, b, d \in B$ and $f, g \in A^{\oplus B}$, we let consider

$$(x)f (x) {}^b g (x) {}^{bd} f,$$

where $dbd = d$. By condition (ii), for $a \in A$, we have $(x)f = a(xbd)f$ where $bd = e$. Thus

$$(x)f (x) {}^b g (x) {}^{bd} f = a(xbd)f (x) {}^b g (x) {}^{bd} f = a(x) {}^{bd} f (x) {}^b g (x) {}^{bd} f. \quad (3)$$

Since A is regular, $A^{\oplus B}$ is regular [8]. Thus we can choose $g = {}^d v$ ($v \in A^{\oplus B}$) such that $f v f = f$ and $v f v = v$. Hence the last term in (3) will be equal to

$$a(x) {}^{bd} f (x) {}^{bd} v (x) {}^{bd} f = a(x) {}^{bd} (f v f) = a(x) {}^{bd} f = (x)f.$$

This implies that $f = f^b g^{bd} f$. On the other hand, by similar procedure as above, we obtain

$$g^d f^{db} g = {}^d v^d f^{dbd} v = {}^d v^d f^d v = {}^d(vfv) = {}^d v = g.$$

Moreover, by condition (iii), we have $P = P_1 b$ or $P = P_1 b d$, where $P_1 \subseteq A^{\oplus B} \times B$. For the next stage of proof, we will only consider $P = P_1 b$ since similar progress can be applied for the other value of P . Therefore there exists a subset $P_2 = P_1 b d$ of $A^{\oplus B} \times B$ such that

$$Pdb \cup P_2 b \cup P = P_1 bdb \cup P_1 bdb \cup P_1 b = P_1 b \cup P_1 b \cup P_1 b = P_1 b = P$$

and

$$P_2 b d \cup P d \cup P_2 = P_1 b d b d \cup P_1 b d \cup P_1 b d = P_1 b d \cup P_1 b d \cup P_1 b d = P_1 b d = P_2.$$

As a result of these above procedure, for every $(f, P, b) \in A \diamond_v B$, there exists $(g, P_2, d) \in A \diamond_v B$ such that

$$\begin{aligned} (f, P, b)(g, P_2, d)(f, P, b) &= (f^b g^{bd} f, Pdb \cup P_2 b \cup P, bdb) = (f, P, b), \\ (g, P_2, d)(f, P, b)(g, P_2, d) &= (g^d f^{db} g, P_2 b d \cup P d \cup P_2, dbd) = (g, P_2, d). \end{aligned}$$

Hence the result. □

Proof of Theorem 2.3:

(i) For arbitrary $f \in A^{\oplus B}$ and for some $P_1 \subseteq A^{\oplus B} \times B$, there exists $m \in \mathbb{N}$ and $(f_1, P_1, b_1) \in A \diamond_v B$ such that

$$(f, \{(1_A, 1_B)\}, 1_B)^m (f_1, P_1, b_1) (f, \{(1_A, 1_B)\}, 1_B)^m = (f, \{(1_A, 1_B)\}, 1_B)^m.$$

By applying (2), we obtain $(f^m f_1^{b_1} (f^m), \{(1_A, b_1)\} \cup P_1 \cup \{(1_A, 1_B)\}, b_1) = (f^m, \{(1_A, 1_B)\}, 1_B)$. Then we get $b_1 = 1_B$, and so $f^m f_1^{b_1} (f^m) = f^m f_1 f^m$. Therefore $f^m f_1 f^m = f^m$, so $A^{\oplus B}$ is π -regular. Also, for $h_1, h_2 \in E(A^{\oplus B})$, since $(h_1, \{(1_A, 1_B)\}, 1_B), (h_2, \{(1_A, 1_B)\}, 1_B) \in E(A \diamond_v B)$, we have

$$\begin{aligned} (h_1, \{(1_A, 1_B)\}, 1_B)(h_2, \{(1_A, 1_B)\}, 1_B) &= \\ (h_2, \{(1_A, 1_B)\}, 1_B)(h_1, \{(1_A, 1_B)\}, 1_B), & \end{aligned}$$

and so $h_1h_2 = h_2h_1$. Hence $A^{\oplus B}$ is strongly π -inverse monoid.

Similarly, for arbitrary $b \in B$ and for some $P_2 \subseteq A^{\oplus B} \times B$, there exists $n \in \mathbb{N}$ and $(f_2, P_2, b_2) \in A \diamond_v B$ such that

$$(\bar{1}, \{(1_A, 1_B)\}, b)^n (f_2, P_2, b_2) (\bar{1}, \{(1_A, 1_B)\}, b)^n = (\bar{1}, \{(1_A, 1_B)\}, b)^n.$$

This implies that $b^m b_2 b^m = b^m$. Thus B is π -regular. In addition, for $e_1, e_2 \in E(B)$, since $(\bar{1}, \{(1_A, 1_B)\}, e_1), (\bar{1}, \{(1_A, 1_B)\}, e_2) \in E(A \diamond_v B)$ and $A \diamond_v B$ is strongly π -inverse monoid, we have $(\bar{1}, \{(1_A, 1_B)\}, e_1) (\bar{1}, \{(1_A, 1_B)\}, e_2) = (\bar{1}, \{(1_A, 1_B)\}, e_2) (\bar{1}, \{(1_A, 1_B)\}, e_1)$, and so $e_1 e_2 = e_2 e_1$. Thus B is strongly π -inverse monoid.

(ii) Let $h \in E(A^{\oplus B})$ and $e \in E(B)$. Then for the corresponding elements $(h, \{(1_A, 1_B)\}, 1_B)$ and $(\bar{1}, \{(1_A, 1_B)\}, e)$ in $E(A \diamond_v B)$, we have

$$(h, \{(1_A, 1_B)\}, 1_B) (\bar{1}, \{(1_A, 1_B)\}, e) = (\bar{1}, \{(1_A, 1_B)\}, e) (h, \{(1_A, 1_B)\}, 1_B),$$

which implies ${}^e h = h$.

(iii) If $f {}^e f = f$, then $(f, \{(1_A, 1_B)\}, e) \in E(A \diamond_v B)$ and $(f, \{(1_A, 1_B)\}, e) (\bar{1}, \{(1_A, 1_B)\}, e) = (\bar{1}, \{(1_A, 1_B)\}, e) (f, \{(1_A, 1_B)\}, e)$ since $(\bar{1}, \{(1_A, 1_B)\}, e) \in E(A \diamond_v B)$ and $A \diamond_v B$ is strongly π -inverse. Therefore we obtain ${}^e f = f$.

(iv) From (i), for every $f \in \text{Reg}(A^{\oplus B})$, there exists a unique $f_1 \in A^{\oplus B}$ (and thus a unique ${}^e f_1 \in A^{\oplus B}$) such that

$$f {}^e f_1 f = f \quad \text{and} \quad {}^e f_1 f {}^e f_1 = {}^e f_1.$$

Then ${}^e (f {}^e f_1 f) = {}^e f$, further ${}^e f {}^e f_1 {}^e f = {}^e f$ and so ${}^e f {}^e f_1 {}^e f = {}^e f$ (since $e \in E(B)$). Additionally, for every $e \in E(B)$, since $f {}^e f_1 \in E(A^{\oplus B})$, by (ii), we have

$${}^e \underbrace{(f {}^e f_1)}_{f_1} = f {}^e f_1 = {}^e (f f_1)$$

and then we get

$${}^e f_1 {}^e f {}^e f_1 = {}^e f_1.$$

Hence both f and ${}^e f$ are inverses of ${}^e f_1$, and so ${}^e f = f$.

(v) Since $A \diamond_v B$ is a strongly π -inverse monoid, for every $(f, P, b) \in A \diamond_v B$, there exists $m \in \mathbb{N}$ and $(f_1, P_1, b_1) \in A \diamond_v B$ such that

$$(f, P, b)^m (f_1, P_1, b_1) (f, P, b)^m = (f, P, b)^m.$$

Then, by (2), we get

$$(f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f \ b^m \ f_1 \ b^m b_1 (f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f), \quad (4)$$

$$((Pb^{m-1} \cup \dots \cup Pb \cup P)b_1 \cup P_1)b^m \cup (Pb^{m-1} \cup \dots \cup Pb \cup P), b^m b_1 b^m) = (f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f, Pb^{m-1} \cup \dots \cup Pb \cup P, b^m). \quad (5)$$

So $b^m b_1 b^m = b^m$ and thus $b^m \in \text{Reg} B$. First components of the equality in (4) give that

$$f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f \ b^m \ f_1 \ b^m b_1 (f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f) = f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f.$$

For simplicity, let us label $f \ b \ f \ b^2 \ f \ \dots \ b^{m-1} \ f$ by $f^{(m)}$. Thus we have

$$f^{(m)} \ b^m \ f_1 \ b^m b_1 (f^{(m)}) = f^{(m)}.$$

By (iii), we get $b^m b_1 (f^{(m)}) = f^{(m)}$ since $b^m b_1 \in E(B)$. Therefore

$$f^{(m)} \ b^m \ f_1 \ f^{(m)} = f^{(m)},$$

which gives $f^{(m)} \in \text{Reg}(A^{\oplus B})$, as required.

(vi) By equality (4), we have

$$((Pb^{m-1} \cup \dots \cup Pb \cup P)b_1 \cup P_1)b^m \cup (Pb^{m-1} \cup \dots \cup Pb \cup P) = Pb^{m-1} \cup \dots \cup Pb \cup P$$

and then

$$(Pb^{m-1} b_1 b^m \cup \dots \cup P b b_1 b^m \cup P b_1 b^m \cup P_1 b^m) \cup (Pb^{m-1} \cup \dots \cup Pb \cup P) = Pb^{m-1} \cup \dots \cup Pb \cup P.$$

Moreover, by (4), for every $(f, P, b) \in A \diamond_v B$, since $b^m b_1 b^m = b^m$, either $P = P_2 b$ or $P = P_2 b^2 \dots$ or $P = P_2 b^{m-1}$ or $P = P_2 b^m$, where $P_2 \subseteq A^{\oplus B} \times B$.

Otherwise, for $P' = Pb^{m-1} \cup \dots \cup Pb \cup P$, there would not be an equality between P' and $(P'b_1 \cup P_1)b^m \cup P'$ which gives a contradiction to $A \diamond_v B$ to be strongly π -inverse monoid.

For the converse part of the proof, let us assume that the monoids A and B satisfy conditions (i)-(vi). By (v), for every $(f, P, b) \in A \diamond_v B$, there exists $m \in \mathbb{N}$, $f_1 \in A^{\oplus B}$ and $b_1 \in B$ such that

$$b^m b_1 b^m = b^m \quad \text{and} \quad f^{(m)} f_1 f^{(m)} = f^{(m)}.$$

By (iv), we have $f^{(m)} b^m f_1 b^m b_1 f^{(m)} = f^{(m)}$ for $b^m b_1 \in E(B)$. Additionally, by using (vi) we obtain the equality

$$(f, P, b)^m (f_1, P_1, b_1) (a, P, b)^m = (a, P, b)^m,$$

which gives the π -regularity of $A \diamond_v B$.

Now we need to show that $E(A \diamond_v B)$ is commutative. Firstly, for arbitrary $(h, P, e) \in E(A \diamond_v B)$, we must prove that $h \in E(A^{\oplus B})$ and $e \in E(B)$. In fact, if $(h, P, e)^2 = (h, P, e)$, then $e^2 = e$, $Pe \cup P = P$ and $h {}^e h = h$. Thus ${}^e h \in E(A^{\oplus B})$ (since $e^2 = e$ and so $e \in E(B)$), and then, by (v), there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} \underbrace{h {}^e h}_{h} e^2 h \dots e^{m-1} h &= h^{(m)} \in \text{Reg}(A^{\oplus B}) \Rightarrow \\ \underbrace{h e^2 h}_{h} \dots e^{m-1} h &= h^{(m)} \in \text{Reg}(A^{\oplus B}) \Rightarrow \\ &\Rightarrow \dots \Rightarrow h e^{m-1} h = h \in \text{Reg}(A^{\oplus B}). \end{aligned}$$

By (iv), for every $h \in \text{Reg}(A^{\oplus B})$ and every $e \in E(B)$, we have ${}^e h = h$. If we apply the function h to two sides of the equation ${}^e h = h$, we obtain $\underbrace{h {}^e h}_h = h^2$, and hence $h^2 = h$ which means $h \in E(A^{\oplus B})$.

Now, for $(h, P, e), (h_1, P_1, e_1) \in E(A \diamond_v B)$, we have $h, h_1 \in E(A^{\oplus B})$ and

$e, e_1 \in E(B)$. By (i) and (ii), we get

$$\begin{aligned} (h, P, e)(h_1, P_1, e_1) &= (h \underbrace{e h_1}_{h_1}, P e_1 \cup P_1, e e_1) \\ &= (h h_1, \{Case 1 \text{ or } Case 2\}, e e_1) \\ &= (h_1 h, \{Case 1 \text{ or } Case 2\}, e_1 e) \\ &= (h_1 \underbrace{e_1 h}_h, P_1 e \cup P, e_1 e) = (h_1, P_1, e_1)(h, P, e). \end{aligned}$$

Case 1: If $e = e_1$, then $P e_1 \cup P_1 = P e \cup P_1$. Since $e \in E(B)$, P must be $P_2 e$, where $P_2 \subseteq A^{\oplus B} \times B$. Indeed if we take $P = P_2 e$ we get $P e \cup P_1 = P_2 e^2 \cup P_2 e = P_2 e \cup P_2 e = P_2 e = P_1 e \cup P$.

Case 2: If $P_1 = P$, then $P e_1 \cup P_1 = P e_1 \cup P$. Now we can take $P = P_2 e$ or $P = P_2 e^2$, so we obtain

$$\begin{aligned} P e_1 \cup P &= P_2 e^2 e_1 \cup P_2 e^2 = P_2 e e_1 e \cup P_2 e \quad (\text{since } E(B) \text{ is commutative}) \\ &= P_2 e \cup P_2 e \quad (\text{since } B \text{ is } \pi\text{-regular}) \\ &= P_2 e = P. \end{aligned}$$

From the other side, we have

$$\begin{aligned} P_1 e \cup P &= P e \cup P = P_2 e e \cup P_2 e = P_2 e \cup P_2 e \quad (\text{since } e \in E(B)) \\ &= P_2 e = P. \end{aligned}$$

We note that this case coincides with the general case of P . In other words, for the case $P_1 \neq P$, we can take $P = P_2 e$ or $P = P_2 e^2$, as well.

Hence the result. \square

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