

APPROXIMATION OF PERIODIC FUNCTIONS IN WEIGHTED LORENTZ SPACES

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ABSTRACT. In the present work, the approximation properties of means of trigonometric Fourier series in weighted Lorentz spaces with weights satisfying the so called Muckenhoupt's A_p conditions are investigated.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbf{T} := [-\pi, \pi]$. A measurable 2π -periodic function $\omega : \mathbf{T} \rightarrow [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x) dx. \quad (1.1)$$

We define the decreasing rearrangement $f_\omega^*(t)$ of $f : \mathbf{T} \rightarrow \mathbb{R}$ with respect to the Borel measure (1.1) by

$$f_\omega^*(t) = \inf \{ \tau \geq 0 : \omega(x \in \mathbf{T} : |f(x)| > \tau) \leq t \}.$$

The weighted Lorentz space $L_\omega^{ps}(\mathbf{T})$ is defined [10, p.20], [1, p.219] as

$$L_\omega^{ps}(\mathbf{T}) = \left\{ f \in \mathbf{M}(\mathbf{T}) : \|f\|_{ps,\omega} = \left(\int_{\mathbf{T}} (f^{**}(t))^s t^{\frac{s}{p}} \frac{dt}{t} \right)^{1/s} < \infty, 1 < p, s < \infty \right\},$$

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where $\mathbf{M}(\mathbf{T})$ is the set of 2π periodic measurable functions on \mathbf{T} and

$$f^{**}(t) = \frac{1}{t} \int_0^t f_{\omega}^*(u) du.$$

If $p = s$, $L_{\omega}^{ps}(\mathbf{T})$ is the weighted Lebesgue space $L_{\omega}^p(\mathbf{T})$ [10, p. 20].

By $E_n(f)_{L_{\omega}^{ps}}$ we denote the best approximation of $f \in L_{\omega}^{ps}(\mathbf{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L_{\omega}^{ps}} = \inf \|f - T_k\|_{ps, \omega},$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$.

The weight functions ω used in the paper belong to the Muckenhoupt class $A_p(\mathbf{T})$ [16] which is defined by

$$\sup \frac{1}{|I|} \int_I \omega(x) dx \left(\frac{1}{|I|} \int_I \omega^{1-p'}(x) dx \right)^{p-1} < \infty, \quad p' = \frac{p}{p-1}, \quad 1 < p < \infty,$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I .

The modulus of continuity of the function $f \in L_{\omega}^{ps}(\mathbf{T})$ is defined [13] as

$$\Omega(f, \delta)_{L_{\omega}^{ps}} = \sup_{|h| \leq \delta} \|A_h f\|_{ps, \omega}, \quad \delta > 0,$$

where

$$(A_h f)(x) := \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt.$$

Whenever $\omega \in A_p(\mathbf{T})$, $1 < p, s < \infty$, the Hardy-Littlewood maximal function of every $f \in L_{\omega}^{ps}(\mathbf{T})$, and therefore the average $A_h f$ belong to $L_{\omega}^{ps}(\mathbf{T})$ [6, Theorem 3]. Thus $\Omega(f, \delta)_{L_{\omega}^{ps}}$ makes sense for every $\omega \in A_p(\mathbf{T})$. The modulus of continuity $\Omega(f, \delta)_{L_{\omega}^{ps}}$ is non-decreasing, non-negative, continuous function such that

$$\lim_{\delta \rightarrow 0} \Omega(f, \delta)_{L_{\omega}^{ps}} = 0, \quad \Omega(f_1 + f_2, \delta)_{L_{\omega}^{ps}} \leq \Omega(f_1, \delta)_{L_{\omega}^{ps}} + \Omega(f_2, \delta)_{L_{\omega}^{ps}}.$$

The modulus of continuity $\Omega(f, \delta)_{L_{\omega}^{ps}}$ is defined in this way, since the space $L_{\omega}^{ps}(\mathbf{T})$ is non-invariant, in general, under the usual shift $f(x) \rightarrow f(x+h)$.

For $0 < \alpha \leq 1$, it is defined the Lipschitz class $Lip(\alpha, L_{\omega}^{ps})$ [9] as

$$Lip(\alpha, L_{\omega}^{ps}) = \{f \in L_{\omega}^{ps}(\mathbf{T}) : \Omega(f, \delta)_{L_{\omega}^{ps}} = O(\delta^{\alpha}), \delta > 0\}$$

and for $r = 1, 2, \dots$ the classes $W_{ps,\omega}^r$, $W_{ps,\omega}^{r,\alpha}$ as

$$W_{ps,\omega}^r = \left\{ f \in L_\omega^{ps}(\mathbf{T}) : f^{(r-1)} \text{ is absolutely continuous and } f^{(r)} \in L_\omega^{ps} \right\}$$

$$W_{ps,\omega}^{r,\alpha} = \left\{ f \in W_{ps,\omega}^r : f^{(r)} \in Lip(\alpha, L_\omega^{ps}) \right\}.$$

Since $L_\omega^{ps}(\mathbf{T}) \subset L^1(\mathbf{T})$ when $\omega \in A_p(\mathbf{T})$, $1 < p, s < \infty$ (see [12, the proof of Prop. 3.3]), we can define the Fourier series of $f \in L_\omega^{ps}(\mathbf{T})$

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \quad (1.2)$$

and the conjugate Fourier series

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx).$$

Here $a_0(f), a_k(f), b_k(f)$, $k = 1, \dots$, are Fourier coefficients of f . Let $S_n(x, f)$, ($n = 0, 1, 2, \dots$) be the n th partial sums of the series (1.2) at the point x , that is,

$$S_n(x, f) := \sum_{k=0}^n A_k(f)(x),$$

where

$$A_0(f)(x) = \frac{a_0}{2}, \quad A_k(f)(x) = a_k(f) \cos kx + b_k(f) \sin kx, \quad k = 1, 2, \dots$$

Let $(p_n)_0^\infty$ be a sequence of positive numbers. We consider two means of the series (1.2) defined by

$$N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} S_m(x, f)$$

and

$$R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^n p_m S_m(x, f),$$

where $P_n = \sum_{m=0}^n p_m$, $p_{-1} = P_{-1} := 0$. In the case $p_n = 1$, $n \geq 0$, both of $N_n(f)(x)$ and $R_n(f)(x)$ are equal to the Cesaro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^n S_m(x, f).$$

The approximation properties of means σ_n in the Lebesgue spaces were investigated by several authors [7, 17, 18, 19]. The generalizations of Quade's results were studied by Mohapatra and Russell [15], Chandra ([2, 3, 4, 5]) and Leindler [14]. In [2], Chandra obtained the estimates for $\|f - N_n(f)\|_p$,

where $1 < p < \infty$. In [3], Chandra gave the estimates for the difference $\|f - R_n(f)\|_p$. In the paper [5], Chandra gave some conditions on the sequence $(p_n)_0^\infty$ and obtained very satisfactory results about approximation by the means $N_n(f)$ and $R_n(f)$ in the Lebesgue spaces. In [8], similar theorems were proved in weighted Lebesgue spaces.

In this work, we generalize the results obtained by Chandra in [5] to the weighted Lorentz spaces for the derivatives of functions. Our main results are the following.

Theorem 1.1. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$ and let $(p_n)_0^\infty$ be a monotonic sequence of positive numbers such that*

$$(n+1)p_n = O(P_n). \quad (1.3)$$

If $f \in W_{ps,\omega}^{r,\alpha}$ and $f^{(r+1)} \in L_\omega^{ps}(\mathbf{T})$ we have

$$\left\| f^{(r)} - N_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O(n^{-\alpha}), \quad n = 1, 2, \dots$$

Theorem 1.2. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$ and let (p_n) be a sequence of positive real numbers satisfying the relation*

$$\sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| = O\left(\frac{P_n}{n+1}\right). \quad (1.4)$$

If $f \in W_{ps,\omega}^{r,\alpha}$ and $f^{(r+1)} \in L_\omega^{ps}(\mathbf{T})$ the estimate

$$\left\| f^{(r)} - R_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O(n^{-\alpha}), \quad n = 1, 2, \dots$$

holds.

If we take $p_n = A_n^{\beta-1}$ ($\beta > 0$), where

$$A_0^\beta = 1, \quad A_k^\beta = \frac{\beta(\beta+1)\dots(\beta+k)}{k!}, \quad k \geq 1,$$

we have

$$N_n \left(f^{(r)} \right) (x) = \sigma_n^\beta \left(f^{(r)} \right) (x) = \frac{1}{A_n^\beta} \sum_{m=0}^n A_{n-m}^{\beta-1} S_m(x, f^{(r)}).$$

Therefore we can estimate the deviation of $f \in W_{ps,\omega}^{r,\alpha}$ from the Cesaro means $\sigma_n^\beta(f^{(r)})$. We formulate this estimate in the following corollary.

Corollary 1.3. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$, $\beta > 0$. If $f \in W_{ps,\omega}^{r,\alpha}$, we have*

$$\left\| f^{(r)} - \sigma_n^\beta \left(f^{(r)} \right) \right\|_{ps,\omega} = O(n^{-\alpha}), \quad n = 1, 2, \dots$$

2. AUXILIARY RESULTS

Lemma 2.1. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$. Then, the estimate*

$$\left\| f^{(r)} - S_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O \left(n^{-\alpha} \right) \quad (2.1)$$

holds for every $f \in W_{ps,\omega}^{r,\alpha}$ and $n = 1, 2, \dots$.

Proof. Let t_n^* ($n = 0, 1, \dots$) be the trigonometric polynomial of the best approximation to $f^{(r)}$ in L_ω^{ps} . Then we have

$$\left\| f^{(r)} - t_n^* \right\|_{ps,\omega} = \inf \left\| f^{(r)} - t_n \right\|_{ps,\omega},$$

where the infimum is taken over all trigonometric polynomials t_n of degree at most n . From Lemma 2.3 of [20], we have

$$\left\| f^{(r)} - t_n^* \right\|_{ps,\omega} = O \left(\Omega \left(f^{(r)}, 1/n \right)_{L_\omega^{ps}} \right).$$

Therefore

$$\left\| f^{(r)} - t_n^* \right\|_{ps,\omega} = O \left(n^{-\alpha} \right).$$

Since the partial sums $S_n(f)$ is uniform bounded in the space $L_\omega^{ps}(\mathbf{T})$ (see [12], [11, Th. 6.6.2], [21, Chap. VI]), we obtain

$$\begin{aligned} \left\| f^{(r)} - S_n \left(f^{(r)} \right) \right\|_{ps,\omega} &\leq \left\| f^{(r)} - t_n^* \right\|_{ps,\omega} + \left\| t_n^* - S_n \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &= \left\| f^{(r)} - t_n^* \right\|_{ps,\omega} + \left\| S_n \left(t_n^* - f^{(r)} \right) \right\|_{ps,\omega} \\ &= O \left(\left\| f^{(r)} - t_n^* \right\|_{ps,\omega} \right) = O \left(n^{-\alpha} \right). \end{aligned}$$

□

Lemma 2.2. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $r \in \mathbb{N}$. Then, the estimate*

$$\left\| S_n \left(f^{(r)} \right) - \sigma_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O \left(n^{-1} \right) \quad (2.2)$$

holds for every $f \in W_{ps,\omega}^{r,1}$, $f^{(r+1)} \in L_\omega^{ps}(\mathbf{T})$ and $n = 1, 2, \dots$.

Proof. If $f^{(r)}$ has the Fourier series

$$f^{(r)}(x) \sim \sum_{k=0}^{\infty} A_k(f^{(r)})(x),$$

then the Fourier series of the conjugate function $\tilde{f}^{(r+1)}(x)$ is

$$\tilde{f}^{(r+1)}(x) \sim \sum_{k=0}^{\infty} k A_k(f^{(r)})(x).$$

On the other hand,

$$\begin{aligned} S_n(f^{(r)})(x) - \sigma_n(f^{(r)})(x) &= \sum_{k=1}^n \frac{k}{n+1} A_k(f^{(r)})(x) \\ &= \frac{1}{n+1} S_n(\tilde{f}^{(r+1)})(x). \end{aligned}$$

Since the partial sums and the conjugate operator is uniform bounded in the space $L_{\omega}^{ps}(\mathbf{T})$ (see [12], [11, Th. 6.6.2], [21, Chap. VI]), we get that

$$\begin{aligned} \left\| S_n(f^{(r)}) - \sigma_n(f^{(r)}) \right\|_{ps,\omega} &= \frac{1}{n+1} \left\| S_n(\tilde{f}^{(r+1)}) \right\|_{ps,\omega} \leq C \frac{1}{n+1} \left\| \tilde{f}^{(r+1)} \right\|_{ps,\omega} \\ &\leq C \frac{1}{n+1} \left\| f^{(r+1)} \right\|_{ps,\omega} = O(n^{-1}) \end{aligned}$$

for $n = 1, 2, \dots$ □

Lemma 2.3 ([5]). *Let (p_n) be a non-increasing sequence of positive numbers. Then,*

$$\sum_{m=1}^n m^{-\alpha} p_{n-m} = O(n^{-\alpha} P_n)$$

for $0 < \alpha < 1$.

3. PROOF OF THE NEW RESULTS

Proof of Theorem 1.1. Let $0 < \alpha < 1$. Since we can write

$$f^{(r)}(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} f^{(r)}(x),$$

we have

$$f^{(r)}(x) - N_n(f^{(r)})(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} \left[f^{(r)}(x) - S_m(f^{(r)})(x) \right].$$

From Lemma 2.1, Lemma 2.3 and condition (1.3) we get

$$\begin{aligned}
\|f^{(r)} - N_n(f^{(r)})\|_{ps,\omega} &\leq \frac{1}{P_n} \sum_{m=0}^n p_{n-m} \|f^{(r)} - S_m(f^{(r)})\|_{ps,\omega} \\
&= \frac{1}{P_n} \sum_{m=1}^n p_{n-m} O(m^{-\alpha}) + \frac{p_n}{P_n} \|f^{(r)} - S_0(f^{(r)})\|_{ps,\omega} \\
&= \frac{1}{P_n} O(n^{-\alpha} P_n) + O\left(\frac{1}{n+1}\right) \\
&= O(n^{-\alpha}).
\end{aligned}$$

Now let $\alpha = 1$. It is easy to see that

$$N_n(f^{(r)})(x) = \frac{1}{P_n} \sum_{m=0}^n p_{n-m} A_m(f^{(r)})(x).$$

From Abel transform,

$$\begin{aligned}
S_n(f^{(r)})(x) - N_n(f^{(r)})(x) &= \frac{1}{P_n} \sum_{m=1}^n (P_n - P_{n-m}) A_m(f^{(r)})(x) \\
&= \frac{1}{P_n} \sum_{m=1}^n \left(\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right) \left(\sum_{k=1}^m k A_k(f^{(r)})(x) \right) \\
&\quad + \frac{1}{n+1} \sum_{k=1}^n k A_k(f^{(r)})(x),
\end{aligned}$$

and therefore

$$\begin{aligned}
\|S_n(f^{(r)}) - N_n(f^{(r)})\|_{ps,\omega} &\leq \frac{1}{P_n} \sum_{m=1}^n \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| \\
&\quad \times \left\| \sum_{k=1}^m k A_k(f^{(r)}) \right\|_{ps,\omega} + \frac{1}{n+1} \left\| \sum_{k=1}^n k A_k(f^{(r)}) \right\|_{ps,\omega}.
\end{aligned}$$

Since

$$S_n(f^{(r)})(x) - \sigma_n(f^{(r)})(x) = \frac{1}{n+1} \sum_{k=1}^n k A_k(f^{(r)})(x),$$

by Lemma 2.2 we obtain

$$\left\| \sum_{k=1}^n k A_k(f^{(r)}) \right\|_{ps,\omega} = (n+1) \|S_n(f^{(r)}) - \sigma_n(f^{(r)})\|_{ps,\omega} = O(1).$$

Therefore we have

$$\begin{aligned}
& \left\| S_n \left(f^{(r)} \right) - N_n \left(f^{(r)} \right) \right\|_{ps,\omega} \\
& \leq \frac{1}{P_n} \sum_{m=1}^n \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| O(1) + O(n^{-1}) \\
& = O\left(\frac{1}{P_n}\right) \sum_{m=1}^n \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| + O(n^{-1}). \quad (3.1)
\end{aligned}$$

By a simple computation, it can be easy to see that

$$\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} = \frac{1}{m(m+1)} \left(\sum_{k=n-m+1}^n p_k - mp_{n-m} \right).$$

This equality implies that

$$\left(\frac{P_n - P_{n-m}}{m} \right)_{m=1}^{n+1}$$

is non-increasing whenever (p_n) is non-decreasing and non-decreasing whenever (p_n) is non-increasing. This shows the equality

$$\sum_{m=1}^n \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| = \left| p_n - \frac{P_n}{n+1} \right| = \frac{1}{n+1} O(P_n).$$

From this and the inequality (3.1), we have

$$\left\| S_n \left(f^{(r)} \right) - N_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O(n^{-1}).$$

Combining the last estimate with (2.1) we get

$$\left\| f^{(r)} - N_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O(n^{-1}).$$

□

Proof of Theorem 1.2. Let $0 < \alpha < 1$. Using definition of $R_n(f^{(r)})(x)$, we can write

$$f^{(r)}(x) - R_n \left(f^{(r)} \right) (x) = \frac{1}{P_n} \sum_{m=0}^n p_m \left[f^{(r)}(x) - S_m \left(f^{(r)} \right) (x) \right].$$

By Lemma 2.1, we obtain

$$\begin{aligned}
\|f^{(r)} - R_n(f^{(r)})\|_{ps,\omega} &\leq \frac{1}{P_n} \sum_{m=0}^n p_m \|f^{(r)} - S_m(f^{(r)})\|_{ps,\omega} \\
&= O\left(\frac{1}{P_n}\right) \sum_{m=1}^n p_m m^{-\alpha} + \frac{p_0}{P_n} \|f^{(r)} - S_0(f^{(r)})\|_{ps,\omega} \\
&= O\left(\frac{1}{P_n}\right) \sum_{m=1}^n p_m m^{-\alpha}. \tag{3.2}
\end{aligned}$$

Using Abel transform, we get

$$\begin{aligned}
\sum_{m=1}^n p_m m^{-\alpha} &= \sum_{m=1}^{n-1} P_m [m^{-\alpha} - (m+1)^{-\alpha}] + n^{-\alpha} P_n \\
&\leq \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} + n^{-\alpha} P_n,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} &= \sum_{m=1}^{n-1} \left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right) \left(\sum_{k=1}^m k^{-\alpha} \right) + \frac{P_n}{n+1} \sum_{m=1}^{n-1} m^{-\alpha} \\
&= O(n^{-\alpha} P_n)
\end{aligned}$$

by condition (1.4). This implies

$$\sum_{m=1}^n p_m m^{-\alpha} = O(n^{-\alpha} P_n).$$

From this and (3.2) we obtain

$$\|f^{(r)} - R_n(f^{(r)})\|_{ps,\omega} = O(n^{-\alpha}).$$

Now let $\alpha = 1$. From Abel transform we get

$$\begin{aligned}
R_n(f^{(r)})(x) &= \frac{1}{P_n} \sum_{m=0}^{n-1} \left[P_m (S_m(f^{(r)})(x)) \right. \\
&\quad \left. - S_{m+1}(f^{(r)})(x) \right] + P_n S_n(f^{(r)})(x) \\
&= \frac{1}{P_n} \sum_{m=0}^{n-1} P_m \left(-A_{m+1}(f^{(r)})(x) \right) + S_n(f^{(r)})(x),
\end{aligned}$$

and therefore

$$R_n \left(f^{(r)} \right) (x) - S_n f^{(r)}(x) = -\frac{1}{P_n} \sum_{m=0}^{n-1} P_m A_{m+1} \left(f^{(r)} \right) (x).$$

Using Abel transform again, we get

$$\begin{aligned} \sum_{m=0}^{n-1} P_m A_{m+1} \left(f^{(r)} \right) (x) &= \sum_{m=0}^{n-1} \frac{P_m}{m+1} (m+1) A_{m+1} \left(f^{(r)} \right) (x) \\ &= \sum_{m=0}^{n-1} \left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right) \left(\sum_{k=0}^m (k+1) A_{k+1} \left(f^{(r)} \right) (x) \right) \\ &\quad + \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1) A_{k+1} \left(f^{(r)} \right) (x). \end{aligned}$$

By considering (2.2) and (1.4) we get

$$\begin{aligned} &\left\| \sum_{m=0}^{n-1} P_m A_{m+1} \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &\leq \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| \left\| \sum_{k=0}^m (k+1) A_{k+1} \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &\quad + \frac{P_n}{n+1} \left\| \sum_{k=0}^{n-1} (k+1) A_{k+1} \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &= \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| (m+2) \left\| S_{m+1} \left(f^{(r)} \right) - \sigma_{m+1} \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &\quad + P_n \left\| S_n \left(f^{(r)} \right) - \sigma_n \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &= O(1) \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| + O \left(\frac{P_n}{n} \right). \end{aligned}$$

This yields

$$\begin{aligned} \left\| R_n \left(f^{(r)} \right) - S_n \left(f^{(r)} \right) \right\|_{L_w^{ps}} &= \frac{1}{P_n} \left\| \sum_{m=0}^{n-1} P_m A_{m+1} \left(f^{(r)} \right) \right\|_{ps,\omega} \\ &= \frac{1}{P_n} O \left(\frac{P_n}{n} \right) = O \left(\frac{1}{n} \right). \end{aligned}$$

Combining this estimate with (2.1), we get

$$\left\| f^{(r)} - R_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O \left(n^{-1} \right).$$

□

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