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ORIGINAL ARTICLE

Optimizing industrial robot selection using novel trigonometric Pythagorean fuzzy normal aggregation operators

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Abstract

The modern world uses an increasing number of robots, notably service robots. Robots will be able to easily manipulate everyday objects in the future, but only if they are paired with planning and decision-making procedures that allow them to comprehend how to complete a task. This research presents new techniques to handling multi-attribute problem solving with trigonometric Pythagorean normal fuzzy numbers. The sine trigonometric Pythagorean fuzzy sets combine the concept of Pythagorean fuzzy sets with sine trigonometric functions to represent uncertainty in decision-making. It is feasible to combine trigonometric Pythagorean fuzzy numbers and normal fuzzy numbers to get trigonometric Pythagorean fuzzy normal numbers. In addition to the fundamental interaction aggregation operators, we define the trigonometric Pythagorean fuzzy normal numbers. The trigonometric Pythagorean fuzzy normal numbers satisfy the following properties: associative, distributive, idempotent, bounded, commutative and monotonicity. Four novel approaches are introduced such as weighted averaging, weighted geometric, generalized weighted averaging and generalized weighted geometric. These operators can be used in the development of a multi-attribute decision-making algorithm. We demonstrate how improved Euclidean and Hamming distances are used in practical situations. For industrial robots, the two most crucial elements are computer science and machine tool technology. The four criteria of weights, orientations, speeds and accuracy may be used to assess robotic systems. They are also more practical, easier to understand, and more adept at identifying the best answer more quickly. The effectiveness and accuracy of the models we are looking at are demonstrated by comparing many existing models with those that have been developed.

Keywords Decision-making · Pythagorean fuzzy set · Interaction aggregation operators · Multi-attribute decision-making

Mathematics Subject Classification 03B52 · 06D72 · 90B50

Introduction and related literature

Decision-makers find it more difficult to select the best course of action when real-world systems get more complicated. Even if choosing from a wide range of options might be challenging, it is still possible to make the optimal decision. Establishing opportunities, objectives, and limitations on their perspectives is difficult for many firms. As a result, while decision making (DM), both people and organizations

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need take into account a number of goals simultaneously. Every day, a variety of topics relating to MADM are discussed. This suggests that we need to boost our DM skills. Numerous researchers have looked at this topic using a range of methodologies. According to Kaplan et al. [1], artificial intelligence (AI) promises to be a major factor in solving the major problem of the future. Margetts et al. [2] state that major economies have adopted significant governmental measures to encourage AI research and development. Apart from posing moral dilemmas, AI will impact democracy and the labor market and pose a danger to civilization. Multidisciplinary technology assessment (TA) is required to evaluate the potential and hazards of this technology. But according to a number of studies, AI is a general-purpose technology that can have a positive influence on society and the economy across a wide variety of industries. Additionally, deep learning technology will enable internet platforms to generate revenue and offer extremely effective social governance tools. Klinger and Rasskazov [3, 4] claim that the future of societal structures may be altered by the use of AI technology in particular commercial and financial fields. When real-world systems change, it might be difficult to decide. Reducing the number of objectives to one is feasible, but it can be difficult. Limiting people's motives, objectives, and points of view proved challenging for many companies. When individuals or groups are making decisions, they must consider several objectives at the same time. Based on practical reasons, DM is therefore unable to make the best decision. In order to determine the optimal choice, DM creates more dependable and useful techniques. Using an interactive aggregating operator as the foundation, Agostini et al. presented a robotic DM architecture [5].

Research background

Nowadays, ambiguity is a part of almost every issue. To cope with uncertainties, there are several different theories of uncertainty, such as the fuzzy set (FS), intuitionistic FS (IFS), Pythagorean FS (PFS), neutrosophic set (NSS), and Fermatean FS (FFS). Decision-makers should be able to get a membership grade (MG), according to the FS. An IFS concept was later introduced by Atanassov, it is noted that each object has two MGs such as positive ζ and negative η and satisfying $0 \leq \zeta + \eta \leq 1$, for $\zeta, \eta \in [0, 1]$. Yager developed the concept of PFSs, which are defined by their MGs and grade of non-memberships (NMGs) with the condition that $\zeta + \eta \geq 1$ to $\zeta^2 + \eta^2 \leq 1$. There has been extensive research and implementation of IFSs and PFSs in various fields. They still have limited skills when it comes to expressing information. As a result, the experts still had difficulties conveying the information in these sets and their associated information. Cuong et al. [6] proposed a notion of picture FS (PiFS) to overcome this problem. Due to these reasons, we must note that PiFS is an expanded version of IFS that accommodates some additional ambiguities. In PiFS, it observed that the MG ζ, neutral η and non-MG ζ with $0 \leq \zeta + \eta + \alpha \leq 1$; for $\zeta, \eta, \alpha \in [0, 1]$. In addition to ensuring that expert opinions are conveyed, such as "yes," "abstain," "no," and "refusal," the PiFS definition will also prevent the acquisition of missing evaluation details and encourage consistency between the evaluation data and the actual decision environment by ensuring that the acquired information is consistent. The concept of PiFS has not been studied widely despite its many applications and studies. A spherical FS (SFS) has been defined by Shahzaib et al. [7] for some aggregating operator (AO) with MADM and cannot handle the picture fuzzy number. SFS and condition that $0 \leq \zeta^2 + \eta^2 + \alpha^2 \leq 1$, where $\zeta, \eta, \alpha \in [0, 1]$. Jin et al. [8] introduced the concept of linguistic spherical fuzzy AOs. The application of SFSs in DM problems was introduced by Rafiq et al. [9]. In DM problem $\zeta^2 + \eta^2 \geq 1$. Senapati et al. [10] proposed the concept of a FFS. The MG and NMG are characterized by $0\zeta^3 + \eta^3 \leq 1$. To investigate DM problems in geometric AOs on interval-valued PFS (IVPFS), Rahman et al. [11] developed an AO on interval-valued PFS. IVPFS with Einstein AO was introduced by Rahman et al. [12] to be an effective (multi attribute group decision making) MAGDM method. Abdullah et al. discussed the concept of artificial intelligent based three-way DM and its applications via Analyzing S-Box Image Encryption Using TOPSIS Method [13] neural network model for surgical approach [14] and double hierarchy linguistic information [15].

Related work

It involves determining the degree to which each option is close to the best response, assigning a score based on that proximity, and then combining the findings. The concept of bipolar FS was suggested by Akram et al. [16] employing TOPSIS (technique for order preference by similarity to ideal solution). Multi-criteria group decision- making (MCGDM) was introduced by Adeel et al. using m-polar fuzzy linguistic TOPSIS [17]. In 2018, Peng et al. [18] discussed the use of TOPSIS and Multi-attribute border approximation area comparison (MABAC) for single-valued NSS. Zhang et al. [19] presented a TOPSIS-based extension of PFS. Hwang et al. [20] provided an explanation of the MADM problems. Complex PFS and its use in pattern recognition were covered by Ullah et al. [21]. Yang et al. [22] described the fuzzy c-number clustering technique. Neutrosophy, or knowledge of the neutral mind, is the difference between FS and IFS. It represents each of the truth grade (TG), indeterminacy grade (IG), and false grade (FG) and has a range of 0 to 1. The NSS approach is a generalization of FS and IVFS. Smarandache et al. presented an interval-valued Pythagorean neutrosophic [23]. By utilizing context analysis [24] and the NSS [25], the single-valued NSS may be utilized to establish a medical diagnosis. Palanikumar et al. [26] have investigated a few algebraic structures and their applications. Yang et al. [27] spoke about interval-valued PFN information with AOs. Using their AO, Palanikumar et al. [28] deals that the novel idea of MADM based on sine trigonometric fermatean normal FS. Ye et al. [29] developed the sine trigonometric AOs using PFS. Recently, Rahim et al. [30] discussed the concept of sine trigonometric (p, q)-quasi rung orthopair AOs and their applications. Abdullah et al. discussed the real life application such as Heterogeneous wireless network [31], deep learning techniques [32]and neural network to select nano materials for nano sensors [33].

Artificial intelligence (AI) and the q-rung orthopair fuzzy logarithmic technique were developed by Petchimuthu et al. [34]. Sugeno Weber triangular norm-based IVFSs are used to improve AI models, according to Gul et al. [35]. Dobrodolac et al. [36] discussed the investigation of possible AI applications. Discussions of recent FSs and related works are necessary. For example, Badi et al. [37] explore the MCDM approach for strategic sustainable development. The assessment of political and ideological education for sustainable development was discussed by Hussain [38]. The idea of Hamacher AOs for PFS and its use in MADM were covered by Asif et al. [39]. Imran [40] discusses the concept of robot selection through Aczel-Alsina Bonferroni Means utilizing IVIFS. Tolga et al. [41] proposed the smart system in hydroponic vertical farming using fuzzy MCDM techniques. Fuzzy Dombi EDAS was proposed by Deveci et al. [42] to assess Metaverse integration of freight fluidity measurement alternatives. Tolga et al. [43] dealt with type-2 Gaussian fuzzy numbers with finite interval values applied to fuzzy TODIM methods. Alali et al. [44] discussion of the TODIM approach for portfolio allocation. The idea of Pythagorean fuzzy soft Einstein AOs and its uses were covered by Ali et al. [45]. Zulqarnain et al. [46] used the TOPSIS technique, which is based on the correlation coefficient, for interval-valued q-rung orthopair fuzzy hypersoft sets in MADM.

Benefits and new features of the suggested approaches

This study will explore the idea of sine trigonometric Pythagorean fuzzy normal number (TPFNN) information in order to better understand it. TPFNN data can be provided via AOs. There are seven sections to this study. PFS and sine trigonometric Pythagorean fuzzy numbers are briefly introduced in Sect. "Preliminaries". In Sect. "Distance measure for sine trigonometric Pythagorean fuzzy normal number", the use of an TPFNN for MADM is covered. In Sect. "Aggregation operators for TPFNN", TPFNN based on MCDM is investigated for different interaction AOs. Section "MADM based on sine trigonometric PFN" discusses, analyzes, and illustrates a method based on TPFNNs. Section 6 contains the conclusions. We can deal with the issue that $0 \preceq (\zeta^t_\Omega(\varepsilon)) + (\eta^f_\Omega(\varepsilon)) > 1$ using the DM technique. A condition that $0 \preceq (\sin \pi/2\zeta^t_\Omega(\varepsilon))^2 + (\sin \pi/2\eta^f_\Omega(\varepsilon))^2 \preceq 1$ was thus added.

The following are the objectives and motivations behind this paper:

1. We have established several algebraic properties of the TPFNN approach, including associativity, distributivity, and idempotent.
2. Hamming distance (HD) and Euclidean distance (ED) are the characteristics of an sine trigonometric Pythagorean fuzzy normal number (TPFNN). To achieve this goal, we need to develop an TPFNN algorithm. By using TPFNN, a normalized decision matrix is determined.
3. To determine an ideal value for sine trigonometric Pythagorean fuzzy normal interaction weighted averaging (TPFNIWA), weighted geometric (TPFNIWG), generalized weighted averaging (TGPFNIWA) and generalized weighted geometric (TGPFNIWG).
4. With the TPFNIWA, TPFNIWG, TGPFNIWA and TGPFNIWG operators in a flexible manner, different ranking results can be derived for alternatives, allowing DMs to choose the ranking result based on their preferences. As a result of its strong advantages, this method is highly flexible.
5. Additionally examined are some of these operators important characteristics, such as boundedness, idempotency, and monotonicity.

Preliminaries

This section presents fundamental concept on the PFS and the sine trigonometric PFS (TPFS) on the universal set Θ.

Definition 2.1 The PFS $\Omega = \left\{\varepsilon, \left\langle \zeta^t_\Omega(\varepsilon), \eta^f_\Omega(\varepsilon)\right\rangle, \varepsilon \in \Theta\right\}$, where $\zeta^t_\Omega : \Theta \to [0, 1]$ and $\eta^f_\Omega : \Theta \to [0, 1]$ represent the MG and NMG of Ω, respectively and $0 \preceq (\zeta^t_\Omega(\varepsilon))^2 + (\eta^f_\Omega(\varepsilon))^2 \preceq 1$. For, $\Omega = \left\langle \zeta^t_\Omega, \zeta^f_\Omega\right\rangle$ is called the Pythagorean fuzzy number (PFN).

Definition 2.2 The FFS $\Omega = \left\{\varepsilon, \left\langle \zeta^t_\Omega(\varepsilon), \eta^f_\Omega(\varepsilon)\right\rangle, \varepsilon \in \Theta\right\}$, where $\zeta^t_\Omega : \Theta \to [0, 1]$ and $\eta^f_\Omega : \Theta \to [0, 1]$ represent the MG and NMG of Ω, respectively and $0 \preceq (\zeta^t_\Omega(\varepsilon))^3 + (\eta^f_\Omega(\varepsilon))^3 \preceq 1$. For, $\Omega = \left\langle \zeta^t_\Omega, \zeta^f_\Omega\right\rangle$ is called the Fermatean fuzzy number (FFN).

Definition 2.3 For any PFNs, $\Omega = \langle \zeta^t, \eta^f\rangle$, $\Omega_1 = \langle \zeta^t_1, \eta^f_1\rangle$ and $\Omega_2 = \langle \zeta^t_2, \eta^f_2\rangle$, ζ^t, η^f denotes MG and NMG, respectively. Then

1. $\Omega_1 \boxplus \Omega_2 = \left(\sqrt{(\zeta^t_1)^2 + (\zeta^t_2)^2 - (\zeta^t_1)^2 \cdot (\zeta^t_2)^2}, (\eta^f_1 \cdot \eta^f_2)\right)$

2. $\Omega_1 \boxtimes \Omega_2 = \left((\zeta_1^t \cdot \zeta_2^t), \sqrt{(\eta_1^f)^2 + (\eta_2^f)^2 - (\eta_1^f)^2 \cdot (\eta_2^f)^2} \right)$
3. $\Gamma \cdot \Omega = \left(\sqrt{1 - \left(1 - (\zeta^t)^2\right)^{\Gamma}}, (\eta^f)^{\Gamma} \right)$
4. $\Omega^{\Gamma} = \left((\zeta^t)^{\Gamma}, \sqrt{1 - \left(1 - (\eta^f)^2\right)^{\Gamma}} \right)$

Definition 2.4 For any FFNs, $\Omega = \langle \zeta^t, \eta^f \rangle$, $\Omega_1 = \langle \zeta_1^t, \eta_1^f \rangle$ and $\Omega_2 = \langle \zeta_2^t, \eta_2^f \rangle$, ζ^t, η^f denotes MG and NMG respectively. Then the basic AO is defined as follows:

1. $\Omega_1 \boxplus \Omega_2 = \left(\sqrt[3]{(\zeta_1^t)^3 + (\zeta_2^t)^3 - (\zeta_1^t)^3 \cdot (\zeta_2^t)^3}, (\eta_1^f \cdot \eta_2^f) \right)$
2. $\Omega_1 \boxtimes \Omega_2 = \left((\zeta_1^t \cdot \zeta_2^t), \sqrt[3]{(\eta_1^f)^3 + (\eta_2^f)^3 - (\eta_1^f)^3 \cdot (\eta_2^f)^3} \right)$
3. $\Gamma \cdot \Omega = \left(\sqrt[3]{1 - \left(1 - (\zeta^t)^3\right)^{\Gamma}}, (\eta^f)^{\Gamma} \right)$
4. $\Omega^{\Gamma} = \left((\zeta^t)^{\Gamma}, \sqrt[3]{1 - \left(1 - (\eta^f)^3\right)^{\Gamma}} \right)$

Definition 2.5 Let $\Omega = \left\langle \zeta^t, \zeta^f \right\rangle$ be a Pythagorean fuzzy number (PFN). We define a sin trigonometric PFN (TPFN) is defined as $\sin \Omega = \left\{ \sin\left(\pi/2 \cdot (\zeta_L^t(x))\right), 1 - \sin(\pi/2 \cdot (1 - \zeta_L^f(x))) \right\}$. It is clear that $\sin \Omega$ is a PFN and condition that $\sin\left(\pi/2 \cdot \zeta_L^t(x)\right) : \mathscr{U} \rightarrow [0,1]$ such that $0 \leq \sin\left(\pi/2 \cdot \zeta_L^t(x)\right) \leq 1$ and $1 - \sin\left(\pi/2 \cdot \left(1 - \zeta_L^f(x)\right)\right) : \mathscr{U} \rightarrow [0,1]$ such that $0 \leq \sin\left(\pi/2 \cdot \zeta_L^t(x)\right) + 1 - \sin\left(\pi/2 \cdot \left(1 - \zeta_L^f(x)\right)\right) \leq 1$. Therefore, $\left\{ \sin\left(\pi/2 \cdot (\zeta_L^t(x))\right), 1 - \sin\left(\pi/2 \cdot (1 - \zeta_L^f(x))\right) \right\}$ is a PFN and $x \in X$, where X is a non-empty set.

Definition 2.6 If $\Omega_1 = \left\langle \zeta_1^t, \eta_1^f \right\rangle$ and $\Omega_2 = \left\langle \zeta_2^t, \eta_2^f \right\rangle$ are any two PFNs and Γ be a positive integer. Then, the interaction AO is defined as

1. $\Omega_1 \boxplus \Omega_2 = \left(\begin{array}{c} \sqrt{(\zeta_1^t)^2 + (\zeta_2^t)^2 - (\zeta_1^t)^2 \cdot (\zeta_2^t)^2}, \\ \sqrt{(\eta_1^f)^2 + (\eta_2^f)^2 - (\eta_1^f)^2 \cdot (\eta_2^f)^2 - (\eta_1^f)^2 \cdot (\zeta_2^t)^2 - (\zeta_1^t)^2 \cdot (\eta_2^f)^2} \end{array} \right)$
2. $\Omega_1 \boxtimes \Omega_2 = \left(\begin{array}{c} \sqrt{(\zeta_1^t)^2 + (\zeta_2^t)^2 - (\zeta_1^t)^2 \cdot (\zeta_2^t)^2 - (\zeta_1^t)^2 \cdot (\eta_2^f)^2 - (\eta_1^f)^2 \cdot (\zeta_2^t)^2}, \\ \sqrt{(\eta_1^f)^2 + (\eta_2^f)^2 - (\eta_1^f)^2 \cdot (\eta_2^f)^2} \end{array} \right)$
3. $\Gamma \cdot \Omega_1 = \left(\sqrt{1 - \left(1 - (\zeta_1^t)^2\right)^{\Gamma}}, \sqrt{\left(1 - (\zeta_1^t)^2\right)^{\Gamma} - \left(1 - (\zeta_1^t + \eta_1^f)^2\right)^{\Gamma}} \right)$
4. $\Omega_1^{\Gamma} = \left(\sqrt{\left(1 - (\eta_1^f)^2\right)^{\Gamma} - \left(1 - (\zeta_1^t + \eta_1^f)^2\right)^{\Gamma}}, \sqrt{1 - \left(1 - (\eta_1^f)^2\right)^{\Gamma}} \right)$

Definition 2.7 Let $\Omega = \left\langle \zeta^t, \eta^f \right\rangle$ be the PFN. Then $S(\Omega) = (\zeta^t)^2 - (\eta^f)^2$, $S(\Omega) \in [-1, 1]$ is called the score function and $H(\Omega) = (\zeta^t)^2 + (\eta^f)^2$, $H(\Omega) \in [0, 1]$ is called the accuracy function.

Definition 2.8 Let $M_1 = (\varrho_1, \sigma_1) \in \mathbb{N}$and $M_2 = (\varrho_2, \sigma_2) \in \mathbb{N}$, then the distance between M_1 and M_2 is $\Delta(M_1, M_2) = \sqrt[2]{(\varrho_1 - \varrho_2)^2 + \frac{1}{2}(\sigma_1 - \sigma_2)^2}$.

Distance measure for sine trigonometric Pythagorean fuzzy normal number

The benefits of aggregating operators (AOs) are covered in this section. In sine trigonometry, the Pythagorean fuzzy normal number is employed. The decision maker has additional options for interpretation when these operators are combined with the WA and WG operators. A brief description of these four operators is given. Boundedness, associativity, monotonicity, idempotency, and commutativity are the primary characteristics of AOs. In this section, $\sphericalangle = \frac{\pi}{2}$ is used.

Definition 3.1 Let $(\varrho, \sigma) \in \mathbb{N}$, $\Omega = \left\langle (\varrho, \sigma); \sin(\sphericalangle \cdot \zeta^t), \sin(\sphericalangle \cdot \eta^f) \right\rangle$ is a TPFNN, where its MG and NMG are defined as $\sin(\sphericalangle \cdot \zeta^t) = \sin(\sphericalangle \cdot \zeta^t) e^{-\left(\frac{x-\varrho}{\sigma}\right)^2}$ and $\sin(\sphericalangle \cdot \eta^f) = 1 - \left(1 - \sin(\sphericalangle \cdot \eta^f)\right) e^{-\left(\frac{x-\varrho}{\sigma}\right)^2}$, $x \in X$ is a non-empty set and $\sin(\sphericalangle \cdot \zeta^t), \sin(\sphericalangle \cdot \eta^f) \in [0, 1]$ and $0 \leq \left(\sin(\sphericalangle \cdot \zeta^t)(x)\right)^2 + \left(\sin(\sphericalangle \cdot \eta^f)(x)\right)^2 \leq 1$.

Definition 3.2 If $\Omega_1 = \left\langle (\varrho_1, \sigma_1); \sin(\sphericalangle \cdot \zeta_1^t), \sin(\sphericalangle \cdot \eta_1^f) \right\rangle$ and $\Omega_2 = \left\langle (\varrho_2, \sigma_2); \sin(\sphericalangle \cdot \zeta_2^t), \sin(\sphericalangle \cdot \eta_2^f) \right\rangle$ are any two TPFNNs. Then the basic operational laws of TPFNNs is defined as follows:

1. $\Omega_1 \boxplus \Omega_2 = \left(\begin{array}{c} (\varrho_1 + \varrho_2, \sigma_1 + \sigma_2); \\ \sqrt{(\sin(\sphericalangle \cdot \zeta_1^t))^2 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \zeta_1^t))^2 \cdot (\sin(\sphericalangle \cdot \zeta_2^t))^2}, \\ \sqrt{\begin{array}{c} (\sin(\sphericalangle \cdot \eta_1^f))^2 + (\sin(\sphericalangle \cdot \eta_2^f))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2 \cdot (\sin(\sphericalangle \cdot \eta_2^f))^2 \\ -(\sin(\sphericalangle \cdot \eta_1^f))^2 \cdot (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \zeta_1^t))^2 \cdot (\sin(\sphericalangle \cdot \eta_2^f))^2 \end{array}} \end{array} \right)$
2. $\Omega_1 \boxtimes \Omega_2 = \left(\begin{array}{c} (\varrho_1 \cdot \varrho_2, \sigma_1 \cdot \sigma_2); \\ \sqrt{\begin{array}{c} (\sin(\sphericalangle \cdot \zeta_1^t))^2 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \zeta_1^t))^2 \cdot (\sin(\sphericalangle \cdot \zeta_2^t))^2 \\ -(\sin(\sphericalangle \cdot \zeta_1^t))^2 \cdot (\sin(\sphericalangle \cdot \eta_2^f))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2 \cdot (\sin(\sphericalangle \cdot \zeta_2^t))^2 \end{array}}, \\ \sqrt{(\sin(\sphericalangle \cdot \eta_1^f))^2 + (\sin(\sphericalangle \cdot \eta_2^f))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2 \cdot (\sin(\sphericalangle \cdot \eta_2^f))^2} \end{array} \right)$
3. $\Gamma \cdot \Omega_1 = \left(\begin{array}{c} (\Gamma \cdot \varrho_1, \Gamma \cdot \sigma_1); \\ \sqrt{1 - \left(1 - (\sin(\sphericalangle \cdot \zeta_1^t))^2\right)^\Gamma}, \\ \sqrt{\left(1 - (\sin(\sphericalangle \cdot \zeta_1^t))^2\right)^\Gamma - \left(1 - (\sin(\sphericalangle \cdot \zeta_1^t) + \sin(\sphericalangle \cdot \eta_1^f))^2\right)^\Gamma} \end{array} \right)$
4. $\Omega_1^\Gamma = \left(\begin{array}{c} (\varrho_1^\Gamma, \sigma_1^\Gamma); \\ \sqrt{\left(1 - (\sin(\sphericalangle \cdot \eta_1^f))^2\right)^\Gamma - \left(1 - (\sin(\sphericalangle \cdot \zeta_1^t) + \sin(\sphericalangle \cdot \eta_1^f))^2\right)^\Gamma} \\ \sqrt{1 - \left(1 - (\sin(\sphericalangle \cdot \eta_1^f))^2\right)^\Gamma} \end{array} \right)$

Using the operator mentioned above, we can now define the ensuing Theorem.

Theorem 3.1 *Let* $\Omega_1 = \left\langle (\varrho_1, \sigma_1); \sin(\sphericalangle \cdot \zeta_1^t), \sin(\sphericalangle \cdot \eta_1^f) \right\rangle$, $\Omega_2 = \left\langle (\varrho_2, \sigma_2); \sin(\sphericalangle \cdot \zeta_2^t), \sin(\sphericalangle \cdot \eta_2^f) \right\rangle$ *and* $\Omega_3 = \left\langle (\varrho_3, \sigma_3); \sin(\sphericalangle \cdot \zeta_3^t), \sin(\sphericalangle \cdot \eta_3^f) \right\rangle$ *be the TPFNNs. Then*

1. $\Omega_1 \boxplus \Omega_2 = \Omega_2 \boxplus \Omega_1$
2. $(\Omega_1 \boxplus \Omega_2) \boxplus \Omega_3 = \Omega_1 \boxplus (\Omega_2 \boxplus \Omega_3)$
3. $\Omega_1 \boxtimes \Omega_2 = \Omega_2 \boxtimes \Omega_1$
4. $(\Omega_1 \boxtimes \Omega_2) \boxtimes \Omega_3 = \Omega_1 \boxtimes (\Omega_2 \boxtimes \Omega_3)$

Proof The appendix contains the proof of the Theorem. □

Definition 3.3 Let $\Omega_1 = \left\langle (\varrho_1, \sigma_1); \sin(\sphericalangle \cdot \zeta_1^t), \sin(\sphericalangle \cdot \eta_1^f) \right\rangle$ and $\Omega_2 = \left\langle (\varrho_2, \sigma_2); \sin(\sphericalangle \cdot \zeta_2^t), \sin(\sphericalangle \cdot \eta_2^f) \right\rangle$ be TPFNNs. Then ED between Ω_1 and Ω_2 is

$$\Delta_{\mathscr{E}}(\Omega_1, \Omega_2) = \sqrt{\begin{array}{c} \left(\frac{1 + (\sin(\sphericalangle \cdot \zeta_1^t))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2}{2} \varrho_1 - \frac{1 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \eta_2^f))^2}{2} \varrho_2 \right)^2 \\ + \frac{1}{2} \left(\frac{1 + (\sin(\sphericalangle \cdot \zeta_1^t))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2}{2} \sigma_1 - \frac{1 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \eta_2^f))^2}{2} \sigma_2 \right)^2 \end{array}} \tag{1}$$

HD between Ω_1 and Ω_2 is

$$\Delta_{\mathscr{H}}(\Omega_1, \Omega_2) = \left(\begin{array}{c} \frac{1 + (\sin(\sphericalangle \cdot \zeta_1^t))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2}{2} \varrho_1 \\ - \frac{1 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \eta_2^f))^2}{2} \varrho_2 \\ + \frac{1}{2} \left(\begin{array}{c} \frac{1 + (\sin(\sphericalangle \cdot \zeta_1^t))^2 - (\sin(\sphericalangle \cdot \eta_1^f))^2}{2} \sigma_1 \\ - \frac{1 + (\sin(\sphericalangle \cdot \zeta_2^t))^2 - (\sin(\sphericalangle \cdot \eta_2^f))^2}{2} \sigma_2 \end{array} \right) \end{array} \right) \tag{2}$$

Theorem 3.2 *Let* $\Omega_1 = \left\langle(\varrho_1, \sigma_1); \sin(\sphericalangle \cdot \zeta_1^t), \sin(\sphericalangle \cdot \eta_1^f)\right\rangle$, $\Omega_2 = \left\langle(\varrho_2, \sigma_2); \sin(\sphericalangle \cdot \zeta_2^t), , \sin(\sphericalangle \cdot \eta_2^f)\right\rangle$ *and* $\Omega_3 = \left\langle(\varrho_3, \sigma_3); \sin(\sphericalangle \cdot \zeta_3^t), , \sin(\sphericalangle \cdot \eta_3^f)\right\rangle$ *be any three TPFNNs. Then*

1. $\Delta_{\mathscr{E}}(\Omega_1, \Omega_2) = 0$ *if and only if* $\Omega_1 = \Omega_2$
2. $\Delta_{\mathscr{E}}(\Omega_1, \Omega_2) = \Delta_{\mathscr{E}}(\Omega_2, \Omega_1)$
3. $\Delta_{\mathscr{E}}(\Omega_1, \Omega_3) \leq \Delta_{\mathscr{E}}(\Omega_1, \Omega_2) + \Delta_{\mathscr{E}}(\Omega_2, \Omega_3)$.

Proof The appendix contains the proof of the Theorem. □

Corollary 3.1 *For any three TPFNNs* $\Omega_1 = \left\langle(\varrho_1, \sigma_1); \sin(\sphericalangle \cdot \zeta_1^t), \sin(\sphericalangle \cdot \eta_1^f)\right\rangle$, $\Omega_2 = \left\langle(\varrho_2, \sigma_2); \sin(\sphericalangle \cdot \zeta_2^t), \sin(\sphericalangle \cdot \eta_2^f)\right\rangle$ *and* $\Omega_3 = \left\langle(\varrho_3, \sigma_3); \sin(\sphericalangle \cdot \zeta_3^t), \sin(\sphericalangle \cdot \eta_3^f)\right\rangle$. *Then*

1. $\Delta_{\mathscr{H}}(\Omega_1, \Omega_2) = 0$ *if and only if* $\Omega_1 = \Omega_2$
2. $\Delta_{\mathscr{H}}(\Omega_1, \Omega_2) = \Delta_{\mathscr{H}}(\Omega_2, \Omega_1)$
3. $\Delta_{\mathscr{H}}(\Omega_1, \Omega_3) \leq \Delta_{\mathscr{H}}(\Omega_1, \Omega_2) + \Delta_{\mathscr{H}}(\Omega_2, \Omega_3)$.

Aggregation operators for TPFNN

The limitations of TPFN are discussed using the aggregation operator for interactions. Next, we introduce new TPFNIWA, TPFNIWG, TGPFNIWA, and TGPFNIWG under TPFN circumstances respectively.

Sine trigonometric PFN interaction weighted averaging (TPFNIWA) operator

Presenting novel operating laws for the TPFN environment based on sine trigonometric functions is the aim of this section. We provide the TPFNIWA.

Definition 4.1 Let $\Omega_j = \left\langle(\varrho_j, \sigma_j); \sin(\sphericalangle \cdot \zeta_j^t), \sin(\sphericalangle \cdot \eta_j^f)\right\rangle$ be the TPFNNs, ξ_j be a weight of Ω_j and $\xi_j \succ 0, \oslash_{j=1}^n \xi_j = 1$. Then TPFNIWA$(\Omega_1, \Omega_2, \ldots, \Omega_n) = \oslash_{j=1}^n \xi_j \Omega_j$.

Theorem 4.1 *Let* $\Omega_j = \left\langle(\varrho_j, \sigma_j); \sin(\sphericalangle \cdot \zeta_j^t), \sin(\sphericalangle \cdot \eta_j^f)\right\rangle$ *be the TPFNNs. Then TPFNIWA* $(\Omega_1, \Omega_2, \ldots, \Omega_n) =$

$$\left(\begin{array}{c} \left(\oslash_{j=1}^n \xi_j \varrho_j, \oslash_{j=1}^n \xi_j \sigma_j\right); \sqrt{\shortmid - \bigcirc_{j=1}^n \left(\shortmid - (\sin(\sphericalangle \cdot \zeta_j^t))^2\right)^{\xi_j}}, \\ \sqrt{\bigcirc_{j=1}^n \left(\shortmid - (\sin(\sphericalangle \cdot \zeta_j^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^n \left(\shortmid - (\sin(\sphericalangle \cdot \zeta_j^t) + \sin(\sphericalangle \cdot \eta_j^f))^2\right)^{\xi_j}} \end{array} \right). \tag{3}$$

Proof The appendix contains the proof of the Theorem. □

Theorem 4.2 *If* $\Omega_j = \left\langle(\varrho_j, \sigma_j); \sin(\sphericalangle \cdot \zeta_j^t), \sin(\sphericalangle \cdot \eta_j^f)\right\rangle$ *be the TPFNNs and* $\Omega_j = \Omega$, *then*
TPFNIWA$(\Omega_1, \Omega_2, \ldots, \Omega_n) = \Omega$, *where* $j = 1, 2, \ldots, n$.

Proof The appendix contains the proof of the Theorem. □

Theorem 4.3 *Let* $\Omega_j = \left\langle(\varrho_{ji}, \sigma_{ji}); \sin(\sphericalangle \cdot \zeta_{ji}^t), \sin(\sphericalangle \eta_{ji}^f)\right\rangle$ $(j = 1, 2, \ldots, n)$; $(i = 1, 2, \ldots, j_i)$ *be a collection of TPFN, where* $\underline{\varrho} = \min \varrho_{ji}$, $\overline{\varrho} = \max \varrho_{ji}$,
$\underline{\sigma} = \max \sigma_{ji}, \overline{\sigma} = \min \sigma_{ji}, \underline{\sin(\sphericalangle \cdot \zeta^t)} = \min\left(\sin(\sphericalangle \cdot \zeta_{ji}^t)\right)$, $\overline{\sin(\sphericalangle \cdot \zeta^t)} = \max\left(\sin(\sphericalangle \cdot \zeta_{ji}^t)\right)$, $\underline{\sin(\sphericalangle \cdot \eta^f)} = \min\left(\sin(\sphericalangle \eta_{ji}^f\right)$, $\overline{\sin(\sphericalangle \cdot \eta^f)} = \max\left(\sin(\sphericalangle \eta_{ji}^f\right)$.
Then,

$$\begin{aligned} &\left\langle(\underline{\varrho}, \underline{\sigma}); \underline{\sin(\sphericalangle \cdot \zeta^t)}, \overline{\sin(\sphericalangle \cdot \eta^f)}\right\rangle \\ &\quad \leq PNSNIVWA(\Omega_1, \Omega_2, \ldots, \Omega_n) \\ &\quad \leq \left\langle(\overline{\varrho}, \overline{\sigma}); \overline{\sin(\sphericalangle \cdot \zeta^t)}, \underline{\sin(\sphericalangle \cdot \eta^f)}\right\rangle. \end{aligned}$$

(boundedness property).

Proof The appendix contains the proof of the Theorem. □

Theorem 4.4 *Let* $\Omega_j = \left\langle(\varrho_{t_{ji}}, \sigma_{t_{ji}}); \sin(\sphericalangle \cdot \zeta_{t_{ji}}^t), \sin(\sphericalangle \cdot \eta_{t_{ji}}^f)\right\rangle$ *and* $W_j = \left\langle(\varrho_{h_{ji}}, \sigma_{h_{ji}}); \sin(\sphericalangle \cdot \zeta_{h_{ji}}^t), \sin(\sphericalangle \cdot \eta_{h_{ji}}^f\right\rangle(j = 1, 2, \ldots, n)$; $(i = 1, 2, \ldots, j_i)$ *be the two families of TPFNIWAs. For any* j, *if there is* $\varrho_{t_{ji}} \leq \sigma_{h_{ji}}$, $\left(\sin(\sphericalangle \cdot \zeta_{t_{ji}}^t)\right)^2 \leq \left(\sin(\sphericalangle \cdot \zeta_{h_{ji}}^t)\right)^2$ *and* $\left(\sin(\sphericalangle \cdot \eta_{t_{ji}}^f)\right)^2 \geq \left(\sin(\sphericalangle \cdot \eta_{h_{ji}}^f)\right)^2$ *or* $\Omega_j \leq W_j$, *then*

$$\begin{aligned} &TPFNIWA\,(\Omega_1, \Omega_2, \ldots, \Omega_n) \\ &\quad \leq TPFNIWA\,(W_1, W_2, \ldots, W_n) \end{aligned}$$

(monotonicity property).

Proof The appendix contains the proof of the Theorem. □

Sine trigonometric PFN interaction weighted geometric (TPFNIWG) operator

This section aims to introduce novel operating laws based on sine trigonometric functions for the TPFN environment. We provide the weighted geometric operator (TPFNIWG) for the sine trigonometric PFN interaction.

Definition 4.2 Let $\Omega_j = \left\langle(\varrho_j, \sigma_j); \sin(\sphericalangle \cdot \zeta_j^t), \sin(\sphericalangle \cdot \eta_j^f)\right\rangle$ be the TPFNNs, ξ_j be a weight of Ω_j. Then TPFNIWG $(\Omega_1, \Omega_2, \ldots, \Omega_n) = \bigcirc_{j=1}^n \Omega_j^{\xi_j}$.

Theorem 4.5 *If* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot \cdot \zeta_j^t), \sin(\lessdot \cdot \eta_j^f)\rangle$ *be the TPFNNs, then TPFNIWG* $(\Omega_1, \Omega_2, \ldots, \Omega_n) =$

$$\left(\begin{array}{c}\left(\bigcirc_{j=1}^n \varrho_j^{\xi_j}, \bigcirc_{j=1}^n \sigma_j^{\xi_j}\right);\\ \sqrt{\bigcirc_{j=1}^n\left(\mathrm{I}-(\sin(\lessdot\cdot\eta_j^f))^2\right)^{\xi_j} - \bigcirc_{j=1}^n\left(\mathrm{I}-(\sin(\lessdot\cdot\zeta_j^t)+\sin(\lessdot\cdot\eta_j^f))^2\right)^{\xi_j}},\\ \sqrt{\mathrm{I}-\bigcirc_{j=1}^n\left(\mathrm{I}-(\sin(\lessdot\cdot\eta_j^f))^2\right)^{\xi_j}}\end{array}\right) \tag{4}$$

Proof The appendix contains the proof of the Theorem. □

Corollary 4.1 *Let* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot \cdot \zeta_j^t), \sin(\lessdot \cdot \eta_j^f)\rangle$ *be the TPFNNs and all are equal with* $\Omega_j = \Omega$. *Prove that TPFNIWG*$(\Omega_1, \Omega_2, \ldots, \Omega_n) = \Omega$.

Similarly, to prove that monotonicity and boundedness properties of TPFNWA.

Sine trigonometric generalized PFNIWA (TGPFNIWA) operator

In this section sine trigonometric generalized PFNIWA operator is defined.

Definition 4.3 Let $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\zeta_j^i, \sin(\lessdot\cdot\eta_j^f)\rangle$ be the TPFNNs, ξ_j be a weight of Ω_j for $j = 1, 2, \ldots, n$. Then, TGPFNIWA$(\Omega_1, \Omega_2, \ldots, \Omega_n) = \left(\oslash_{j=1}^n \xi_j \Omega_j^2\right)^{1/\Gamma}$.

Theorem 4.6 *Let* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\zeta_j^i, \sin(\lessdot\cdot\eta_j^f)\rangle$ *be the TPFNNs then, TGPFNIWA* $(\Omega_1, \Omega_2, \ldots, \Omega_n) =$

$$\left(\begin{array}{c}\left(\left(\oslash_{j=1}^n \xi_j\varrho_j^2\right)^{1/\Gamma}, \left(\oslash_{j=1}^n \xi_j\sigma_j^2\right)^{1/\Gamma}\right);\\ \left(\sqrt{\mathrm{I}-\bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\zeta_j^t))^2\right)^2\right)^{\xi_j}}\right)^{1/\Gamma},\\ \left(\sqrt{\bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\zeta_j^t))^2\right)^2\right)^{\xi_j} - \bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\zeta_j^t)+\sin(\lessdot\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^{1/\Gamma}\end{array}\right). \tag{5}$$

Proof The appendix contains the proof of the Theorem. □

Remark 4.1 If $\Gamma = 1$, then TGPFNIWA changes to TPFNIWA.

Corollary 4.2 *Let* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\cdot\eta_j^f)\rangle$ *be the TPFNNs and all are equal with* Ω. *Prove that TGPFNIWA* $(\Omega_1, \Omega_2, \ldots, \Omega_n) = \Omega$.

Sine trigonometric generalized PFNIWG (TGPFNIWG) operator

Definition 4.4 Let $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\zeta_j^i, \sin(\lessdot\cdot\eta_j^f)\rangle$ be TPFNNs, ξ_j be a weight of Ω_j. Then, TGPFNIWG $(\Omega_1, \Omega_2, \ldots, \Omega_n) = \frac{1}{\Gamma}\left(\bigcirc_{j=1}^n (\Gamma\Omega_j)^{\xi_j}\right)$.

Theorem 4.7 *Let* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\cdot\zeta_j^i, \sin(\lessdot\cdot\eta_j^f)\rangle$ *be the collection of TPFNNs. Then*

$$TGPFNIWG(\Omega_1, \Omega_2, \ldots, \Omega_n) = \left(\begin{array}{c}\left(\frac{1}{\Gamma}\bigcirc_{j=1}^n (\Gamma\varrho_j)^{\xi_j}, \frac{1}{\Gamma}\bigcirc_{j=1}^n (\Gamma\sigma_j)^{\xi_j}\right);\\ \left(\sqrt{\begin{array}{c}\bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\eta_j^f))^2\right)^2\right)^{\xi_j} -\\ \bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\zeta_j^t)+\sin(\lessdot\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}\end{array}}\right)^{1/\Gamma},\\ \left(\sqrt{\mathrm{I}-\bigcirc_{j=1}^n\left(\mathrm{I}-\left((\sin(\lessdot\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^{1/\Gamma}\end{array}\right) \tag{6}$$

Proof The appendix contains the proof of the Theorem. □

Remark 4.2 If $\Gamma = 1$, then TGPFNIWG becomes TPFNIWG.

Corollary 4.3 *Let* $\Omega_j = \langle(\varrho_j, \sigma_j); \sin(\lessdot\cdot\zeta_j^t), \sin(\lessdot\cdot\eta_j^f)\rangle$ *be the finite collection of TPFNNs. Then TGPFNIWG* $(\Omega_1, \Omega_2, \ldots, \Omega_n) = \Omega$.

MADM based on sine trigonometric PFN

Using the following algorithm, it can be seen that the TOPSIS-selected alternative does not match the negative ideal solution at a minimum. Let $\Xi = \{A, B, \ldots, \Xi_n\}$ is a n-alternative, $A = \{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ is a m-attribute, $w = \{\xi_1, \xi_2, \ldots, \xi_m\}$ is a weights and $\Xi_{ij} = \langle(\varrho_{ij}, \sigma_{ij}); \sin(\lessdot \cdot \zeta^t)_{ij}, \sin(\lessdot \cdot \eta^f)_{ij}\rangle$ be the TPFNN of alternative Ξ_j in attribute κ_j. We consider $\sin(\lessdot \cdot \zeta^t)_{ij}, \sin(\lessdot\zeta^f_{ij} \in [0, 1]$ and $0 \leq (\sin(\lessdot \cdot \zeta^t)_{ij}(\varepsilon))^2 + (\sin(\lessdot\zeta^f_{ij}(\varepsilon))^2 \leq 1$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. For, the decision matrix $\Delta = (\Xi_{ij})_{n\times m}$. A important principle of optimization and DM, especially in the context of MADM, is the necessity that the total of a weight vector's components equal one. In order to enable a meaningful and objective aggregation of criteria, this constraint often referred to as normalization makes sure that the weights allocated to different criteria are proportionate and comparable. The proportional importance or contribution of each criterion to the final decision is reflected in the weights that decision-makers give to the various criteria. In essence, decision-makers are expressing their preferences proportionately by normalizing the weights to add up to one. Each criteria's weight can be understood as the percentage of influence it has over the decision-making process. To obtain a decision, the following algorithm is applied. In addition to MADM, fuzzy information measures can be applied to image segmentation, pattern recognition, clustering analysis, and pattern identification. Occasionally, the outcomes of the two activities are the same. To illustrate the benefits and usefulness of our models, we compare them to many existing ones. a comparison of the suggested models with other models that are currently in use. Because of its worth and advantages, it turns out to be beneficial. We employ the four methods due to the previously described situations.

Algorithm

The optimal choice is determined by this method.

Step 1: Table 1 presents requested data as TPFNs, with a decision matrix to evaluate collection and each alternative.

Step 2: Compute a normalized decision values. Now, $\Delta = (\Xi_{ij})_{n\times m}$ into $\overrightarrow{\Delta} = (\overrightarrow{\Xi_{ij}})_{n\times m}$; put $\overrightarrow{\Xi_{ij}} = \Big\langle(\overrightarrow{\varrho_{ij}}, \overrightarrow{\sigma_{ij}}); \overrightarrow{\sin(\lessdot \cdot \zeta^t_{ij})}, \overrightarrow{\sin(\lessdot \cdot \eta^f_{ij})}\Big\rangle$ and

$$\overrightarrow{\varrho_{ij}} = \frac{\varrho_{ij}}{\sup_j(\varrho_{ij})}, \overrightarrow{\sigma_{ij}} = \frac{\sigma_{ij}}{\sup_j(\sigma_{ij})} \cdot \frac{\sigma_{ij}}{\varrho_{ij}},$$
$$\overrightarrow{\sin(\lessdot \cdot \zeta^t_{ij})} = \sin(\lessdot \cdot \zeta^t_{ij}), \overrightarrow{\sin(\lessdot \cdot \eta^f_{ij})} = \sin(\lessdot \cdot \eta^f_{ij}). \tag{7}$$

Step 3: The TPFN for interacting AOs and the attribute e_j in Ξ_j,

$\overrightarrow{\Xi_{ij}} = \Big\langle(\overrightarrow{\varrho_{ij}}, \overrightarrow{\sigma_{ij}}); \overrightarrow{\sin(\lessdot \cdot \zeta^t_{ij})}, \overrightarrow{\sin(\lessdot \cdot \eta^f_{ij})}\Big\rangle$ is aggregated into

$$\overrightarrow{\Xi_j} = \Big\langle(\overrightarrow{\varrho_j}, \overrightarrow{\sigma_j}); \overrightarrow{\sin(\lessdot \cdot \zeta^t_j)}, \overrightarrow{\sin(\lessdot \cdot \eta^f_j)}\Big\rangle. \tag{8}$$

Step 4: Calculate the positive and negative values as

$$\overrightarrow{\Xi^+} = \Big\langle \sup_{1\leq i\leq n} \overrightarrow{\varrho_{ij}}, \inf_{1\leq i\leq n} \overrightarrow{\sigma_{ij}}, 1, 0\Big\rangle,$$
$$\overrightarrow{\Xi^-} = \Big\langle \inf_{1\leq i\leq n} \overrightarrow{\varrho_{ij}}, \sup_{1\leq i\leq n} \overrightarrow{\sigma_{ij}}, 0, 1\Big\rangle \tag{9}$$

Step 5: Determine the HDs between the options with ideal values that are both positive and negative in order to calculate the value of each option as

$$\Delta^+_j = \Delta_{\mathscr{H}}\Big(\overrightarrow{\Xi_i}, \overrightarrow{\Xi^+}\Big);\ \Delta^-_j = \Delta_{\mathscr{H}}\Big(\overrightarrow{\Xi_i}, \overrightarrow{\Xi^-}\Big) \tag{10}$$

Step 6: The following formula may be used to determine the relative closeness values are

$$\Delta^*_j = \frac{\Delta^-_j}{\Delta^+_j + \Delta^-_j} \tag{11}$$

Step 7: The output yields is $\sup \Delta^*_j$.

The MADM process is represented in a widely used trends in Fig. 1. As illustrated in Fig. 1, this structure usually comprises the following steps: defining criteria, assigning weights to these criteria, and calculating the HD, ED, and score values to identify the optimal choice.

Selection tolls of industrial robots

Recent years have seen a significant evolution in the definition of industrial robot systems. Industrial robots can be understood by examining their evolution from origin to the present. As machines that performed repetitive and relatively static movements, industrial robots were the first robots. Today, the ability to distinguish industrial robots from service robots and how to define their work areas has become more complex as technology advances. Based on their intended applications, robots are classified into industrial and service robots in the World Robotics Report 2021. Service robots perform useful tasks for people or equipment but do not perform industrial automation functions. As a result, industrial robots are no longer restricted to safety zones because robots and humans often share space and tasks within the industry. The use of service robots in industrial settings will increase as they become more common. A collaborative robot is a key component of Industry 4.0. The development of industrial

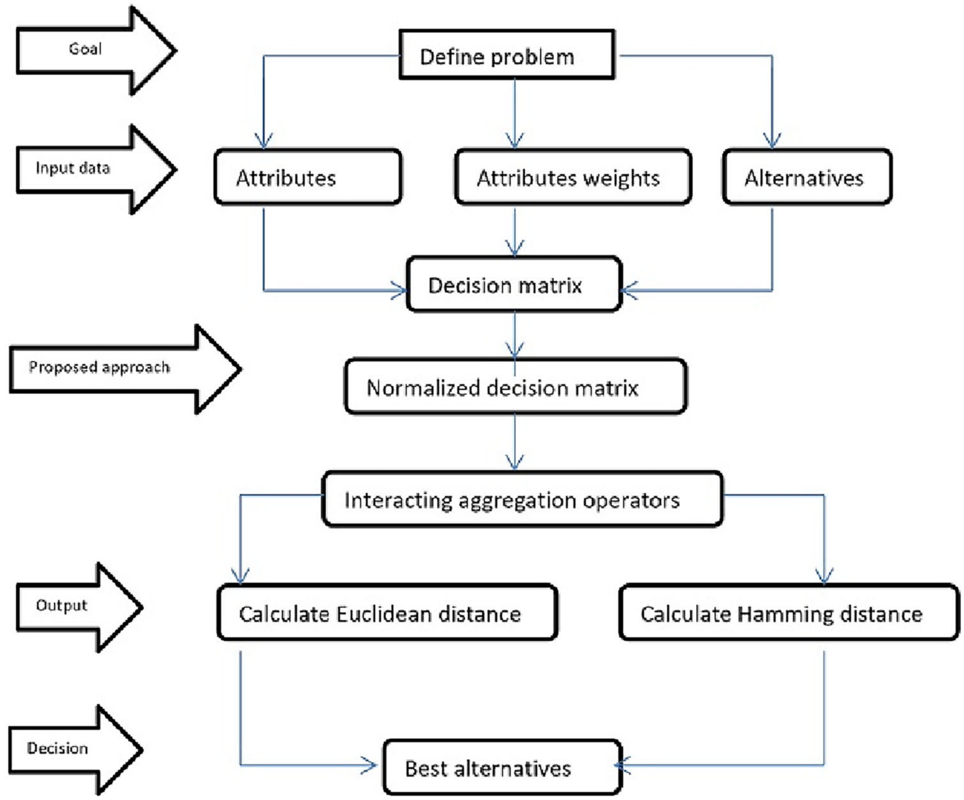

Fig. 1 Algorithm for the proposed methods

robot systems and their path toward intelligent automation solutions utilizing human interaction would not have led to the topic of collaborative robotics today. Assembling a constantly moving assembly line requires industrial robots to automate intensive production tasks. All worker tasks and processes revolve around these large, heavy robots at fixed positions within an industrial plant. Industrial robot characteristics will differ in terms of manufacturing, needs, and the environment in which they are to be used. It is defined in ISO 8373:2012 that an industrial robot is a multi-functional, re-programmable, and automatically controlled manipulator that can be fixed or mobile in industrial automation applications and programmable in three or more axes. Industrial robots are usually not humanoid in form despite their ability to reproduce human movement and behavior. The mobile robot installations are anticipated to increase each year, according to the World Robotics 2021 report.

This section illustrates the application of the developed approaches through a practical MADM problem. Robots are generally thought of as pieces of machinery that resemble humans. An operator of a plant will think of a robot when they think of productivity and assembly. Robots are often defined based on their application, for example, handling robots, palletizing robots, packaging robots, etc., even within this specific definition of machinery. Among the five types of robotics, four are Cartesian, two are cylindrical, one is SCARA, and the other is 6-axis. Different industrial robots are best suited for different applications based on specific attributes. Speed, size, and workspace are their main differences. If a machine designer knows each of the five types of robots' operating aspects, they can select the best robot for their process.

1. Cartesian robot ($\mathfrak{O}_a$): Cartesian robot with XYZ coordinates that moves three axes to a specific location in its workspace based on XYZ coordinates. It can be extremely beneficial to engineers to use Cartesian robots when automating warehouses, improving current systems, or solving unique design challenges. We will examine some of the applications where Cartesian robots offer an advantage over other automated systems, along with motor controllers and drivers. There is a few of Cartesian robots we use daily. In both cases, the mechanics of moving large parts of a multi-story building are the same, regardless if you use a two-axis crane or a 3D printer in a much smaller area. In addition to their extensive range of applications, Cartesian robots offer excellent returns on investment, catering to pick and place processes, labeling, measuring, and other processes that require speed and precision. Cartesian robot systems help to automate these procedures, making an operation that would otherwise require three to four people into one that requires just one, or maybe even none, depending on the complexity of the system, such as peripheral AI.

Industrial robots are typically Cartesian in design. Due to their ease of use and programming, plant operators often use this type of controller. Cartesian elements allow the robot to move linearly in a cube-shaped workspace. This is ideal for pick-and-place applications and can be as small as 100 mms or as large as tens of meters long. A popular reason for choosing these robots is their versatility. However, Cartesian robots can be complicated to assemble. Its flexibility in configuration allows plant operators to meet specific application requirements, making it the most popular robot design.

2. Cylindrical robot ($\mathfrak{O}_b$): An orthogonal slide and a rotary axis comprise the cylindrical robot's construction. Points on a cylinder are traced along three axes. Cartesian designs have a smaller work envelope than cylindrical designs. In the same way as a Cartesian robot, a cylindrical robot has an axis of motion. A cylindrical robot has two types of actuators: linear actuators and rotary actuators. It is space-efficient for machine designers to design machines with cylindrical work envelopes. Because it can be positioned in the middle of the workspace, the robot can work anywhere. Materials that must be picked up, rotated, and placed are best suited for cylindrical robots. It is easy to install and use the solutions since they come as complete packages.

   In a cylindrical coordinate system, point positions are specified by how far they are from a reference axis, how far they are from a reference plane perpendicular to the axis, and how far they are from a chosen reference direction. If the point faces a positive or negative reference plane, the latter distance will be positive or negative. All three coordinates of the system begin at zero, which is the system's origin. An intersection of an axis and a reference plane is found here. In contrast to the polar axis, which lies in the reference plane and starts at the origin, the cylindrical axis is called the longitudinal axis or the longitudinal axis instead of the polar axis. A cylindrical coordinate system is formed by the axes of these robots. In addition to assembly operations, this tool can also be used to handle machine tools, weld spots, and handle die-casting machines. Up and down movements are made by the main arm during operations. Using a cylinder built into the arm, the robot can perform this motion. Motors and gears produce rotation in cylinder robots, while pneumatic cylinders produce up- and down-movement.

3. SCARA robot ($\mathfrak{O}_c$): This robot type has been designed to speed up and improve the steps involved in assembly or picking and placing items from one location to another. As a result, it is often used in conjunction with FlexiBowl. The FlexiBowl robots are also used in other industrial fields where bulk components of all sizes need to be fed to manufacturers; in these areas, FlexiBowl has proven to be more efficient than other parts feeders. In addition to its ability to adjust movement on the horizontal plane and maintain high rigidity in the vertical direction, SCARA's kinematics make this robot ideal for assembly tasks requiring close tolerances, such as inserting shafts into holes. SCARA's mechanical design is generally quite strong and can withstand shocks and collisions without problems. As SCARA robots move vertically, the elbows and shoulders are vertically joined. Gravity has little effect by downloading the robot to the ground, allowing this machine to be used in perforations with high pressures.

   A SCARA robot is ready to use once the end-of-arm tooling is installed. Cylindrical robots have same work envelopes to arc-shaped or radius-shaped robots, but they have a broader range of motion. The applications of SCARA robots are similar to those of cylindrical and Cartesian robots. However, SCARA robots can move faster than their counterparts. The small work area of these devices makes them common in biomedical applications. SCARAs seem like the best solution since they are easy to integrate, but Cartesians are more common because they are more customizable. SCARA robots generally operate at higher speeds and with optional features like clean-room specifications. When speed and accuracy are required for pick-and-place or assembly operations, the SCARA robot is commonly used. Tolerances of less than 10 microns are currently achievable by SCARA Robots. Six-axis robots have tolerances of 20 microns. Additionally, SCARA Robots have a compact layout, which allows them to be relocated to temporary or remote locations more easily. SCARA is designed to suit applications with a smaller area of operation and limited floor space.

4. Six-axis robot ($\mathfrak{O}_d$): Six-axis robots are also all-in-one robots. Even though 6-axis robots can sometimes look like toy cars, they often work on an assembly line, putting seats in cars. These robots can pick up and move materials between planes like human arms. The other types of robots cannot easily pick up parts from tables and place them in cupboards, for example. In addition to being quick and available in complete solutions like SCARAs, 6-axis robots are more complex to program as they move in six directions. Depending on the size and speed of the robots, amusement park rides could be simulated if they were fitted with roller coaster seats. They are one of the five largest types of robots, so designers select them to compensate for the loss of space because they can make movements other robots cannot. In addition to having an additional range of motion, six-axis robots can perform a wider range of robotic tasks. Each axis represents an independent motion as a robot arm moves to a programmed point. A six-axis robot can perform the following movements on each axis: Industrial robots have one axis at their base called axis one. Industrial robots can move their arms 360 degrees from their center using this axis. A robot can move an

object along a straight line using this method. With this motion, an object can be lifted, moved sideways, up and down, and set down along the x or y axes. The third axis of a robot provides it with the capability of raising and lowering the upper arm, allowing it to reach higher in the air. Robots can move parts along the three axes of x, y and z simultaneously, making it easier for them to pick up parts. An object's orientation can be changed by rolling on Axis four, which is responsible for controlling the movements of the robot EOAT. Pitch and yaw movements are controlled by axis five. End effectors are moved up and down during pitch movements. End effectors move left and right with yaw movements. This axis controls an entire 360-degree rotation of the wrist. By rolling, pitching, and yawing movements, industrial robots can change a part's orientation in each of the three planes.

5. Delta robot ($\mathfrak{O}_e$): Each motor shaft is attached to one arm, which must be perpendicular to the shaft's rotational axis. Delta robots typically have three servo motors with high torque mounted to a rigid frame. Parallelogram-shaped connecting rods prevent arms from twisting. An arm is attached to a platform, and a platform is attached to each arm. At both ends of each parallel rod, there are usually ball joints that allow the rod to move freely. It is possible to add an end effector and various other options to the bottom platform, including motors for adding additional degrees of freedom. In order to counteract the increased leverage applied by the payload in the shaft, servo motors need to have high torque. The motors are mounted on the frame, so movement is translated into the arms. A basic and established subject in robotics, kinematics is concerned with the relationship between the joint coordinates of a robot and the manner in which it is arranged spatially. For delta robot kinematics, there are forward kinematics and inverse kinematics. The purpose of inverse kinematics is to determine the appropriate angles $(\theta_1, \theta_2, \theta_3)$ for each of the three arms so that the end effectors can be positioned at their desired positions (X, Y, Z). A forward kinematic system determines the position of an end effector (X, Y, Z) given the angle of the arms $(\theta_1, \theta_2, \theta_3)$.
It is the most expensive and fastest type of robot out of the five. They can achieve very high speeds because of their dome-shaped work envelope. Robots with delta shape are best suited for picking-and-placing operations and transferring parts on and off conveyor belts, such as moving parts from one belt to another. Their use is more complicated than the 6-Axis or SCARA robots, but they are also available as complete solutions for machine designers. Speed and precision are the main advantages of Delta robots.

Table 1 Decision information

	e_1	e_2
$\mathfrak{O}_a$	$\langle(0.9, 0.85); 0.8, 0.35\rangle$	$\langle(0.95, 0.75); 0.7, 0.35\rangle$
$\mathfrak{O}_b$	$\langle(0.85, 0.75); 0.7, 0.56\rangle$	$\langle(0.9, 0.85); 0.7, 0.45\rangle$
$\mathfrak{O}_c$	$\langle(0.8, 0.7); 0.73, 0.63\rangle$	$\langle(0.75, 0.7); 0.24, 0.84\rangle$
$\mathfrak{O}_d$	$\langle(0.75, 0.65); 0.45, 0.56\rangle$	$\langle(0.65, 0.6); 0.45, 0.7\rangle$
$\mathfrak{O}_e$	$\langle(0.85, 0.8); 0.49, 0.63\rangle$	$\langle(0.8, 0.75); 0.24, 0.56\rangle$

Table 2 Decision information

	e_3	e_4
$\mathfrak{O}_a$	$\langle(0.8, 0.75); 0.56, 0.85\rangle$	$\langle(0.8, 0.7); 0.77, 0.87\rangle$
$\mathfrak{O}_b$	$\langle(0.85, 0.8); 0.49, 0.56\rangle$	$\langle(0.85, 0.75); 0.7, 0.49\rangle$
$\mathfrak{O}_c$	$\langle(0.7, 0.55); 0.66, 0.35\rangle$	$\langle(0.7, 0.65); 0.91, 0.35\rangle$
$\mathfrak{O}_d$	$\langle(0.9, 0.75); 0.77, 0.45\rangle$	$\langle(0.85, 0.8); 0.7, 0.49\rangle$
$\mathfrak{O}_e$	$\langle(0.8, 0.7); 0.7, 0.56\rangle$	$\langle(0.9, 0.85); 0.94, 0.77\rangle$

To decide which of the five robot types to use in their processes, designers need to consider the following: load (e_1), orientation (e_2), speed (e_3) and precision (e_4). By analyzing these factors, they can determine what robot type will be most effective and efficient and weights are $w = \{0.4, 0.3, 0.2, 0.1\}$.

Step 1: The weight vector illustrates the relative worth of each criterion as determined by a group of experts. Now, let us create a decision matrix for each alternative. The decision matrix for evaluating robot information is presented in Tables 1 and 2.

By taking the weight of the experts, that is $w = \{0.4, 0.3, 0.2, 0.1\}$.

Step 2: The following is an example of a normalized decision matrix. By Eq. (7), Tables 3 and 4 represents the normalized decision values.

By Eq. (8), It is possible to derive a aggregated normalized decision values are summarized in Tables 5 and 6.

Step 3: To use the TPFNIWA operator to gather data on each expert's alternative. Table 7 displays the results, while Eq. 9 provides an understanding of the entropy. We obtained the values.

Step 4: The ideal values are $\overrightarrow{\Xi^+} = \langle(0.9667, 0.6979), 1, 0\rangle$ and $\overrightarrow{\Xi^-} = \langle(0.8257, 0.8608), 0, 1\rangle$

Step 5: By Eq. (10), we can get the HD of each alternative are

$$\Delta_1^+ = 0.2975, \Delta_2^+ = 0.3052, \Delta_3^+ = 0.3364,$$
$$\Delta_4^+ = 0.3351, \Delta_5^+ = 0.3040.$$

Table 3 Normalized decision values

	e_1	e_2
$\mathfrak{O}_a$	$\langle(0.9, 0.85); 0.2538, 0.084\rangle$	$\langle(0.95, 0.75); 0.1674, 0.084\rangle$
$\mathfrak{O}_b$	$\langle(0.85, 0.75); 0.1674, 0.1341\rangle$	$\langle(0.9, 0.85); 0.1674, 0.3342\rangle$
$\mathfrak{O}_c$	$\langle(0.8, 0.7); 0.2701, 0.1508\rangle$	$\langle(0.75, 0.7); 0.3813, 0.2004\rangle$
$\mathfrak{O}_d$	$\langle(0.75, 0.65); 0.3342, 0.1341\rangle$	$\langle(0.65, 0.6); 0.3342, 0.1674\rangle$
$\mathfrak{O}_e$	$\langle(0.85, 0.8); 0.1174, 0.1508\rangle$	$\langle(0.8, 0.75); 0.3813, 0.1341\rangle$

Table 4 Normalized decision values

	e_3	e_4
$\mathfrak{O}_a$	$\langle(0.8, 0.75); 0.1341, 0.892\rangle$	$\langle(0.8, 0.7); 0.1839, 0.2375\rangle$
$\mathfrak{O}_b$	$\langle(0.85, 0.8); 0.1174, 0.1341\rangle$	$\langle(0.85, 0.75); 0.1674, 0.1174\rangle$
$\mathfrak{O}_c$	$\langle(0.7, 0.55); 0.2862, 0.084\rangle$	$\langle(0.7, 0.65); 0.2169, 0.084\rangle$
$\mathfrak{O}_d$	$\langle(0.9, 0.75); 0.1839, 0.3342\rangle$	$\langle(0.85, 0.8); 0.1674, 0.1174\rangle$
$\mathfrak{O}_e$	$\langle(0.8, 0.7); 0.1674, 0.1341\rangle$	$\langle(0.9, 0.85); 0.2212, 0.1839\rangle$

Table 5 Aggregated normalized decision values

	e_1	e_2
$\mathfrak{O}_a$	$\langle(1, 0.9444); 0.2538, 0.084\rangle$	$\langle(1, 0.6966); 0.1674, 0.084\rangle$
$\mathfrak{O}_b$	$\langle(0.9444, 0.7785); 0.1674, 0.1341\rangle$	$\langle(0.9474, 0.9444); 0.1674, 0.3342\rangle$
$\mathfrak{O}_c$	$\langle(0.8889, 0.7206); 0.2701, 0.1508\rangle$	$\langle(0.7895, 0.7686); 0.3813, 0.2004\rangle$
$\mathfrak{O}_d$	$\langle(0.8333, 0.6627); 0.3342, 0.1341\rangle$	$\langle(0.6842, 0.6516); 0.3342, 0.1674\rangle$
$\mathfrak{O}_e$	$\langle(0.9444, 0.8858); 0.1174, 0.1508\rangle$	$\langle(0.8421, 0.8272); 0.3813, 0.1341\rangle$

and

$$\Delta_1^- = 0.3603, \Delta_2^- = 0.3526, \Delta_3^- = 0.3214,$$
$$\Delta_4^- = 0.3227, \Delta_5^- = 0.3538.$$

Step 6: By Eq. (11), we get the relative closeness values are

$$\Delta_1^* = 0.5477, \Delta_2^* = 0.5360, \Delta_3^* = 0.4886, \Delta_4^* = 0.4905, \Delta_5^* = 0.5378.$$

Step 7: Ranking of alternatives are $\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_b \succ \mathfrak{O}_d \succ \mathfrak{O}_c$. Therefore, based on the computational method described above, we may conclude that the alternative $\mathfrak{O}_a$ is indeed the best choice among the alternatives. As a result, it is highly advised that $\mathfrak{O}_a$ be chosen. The best outcomes with the most value were achieved for each phase. The proposed AOs were evaluated using the available technologies to establish their superiority and validity. The proposed AO performs better in terms of accuracy and dependability than the current methodology. We proposed a novel method for determining the optimum solution to the MADM problem.

Table 7 TPFNIWA operator

	$\vec{\mathfrak{H}}$
$\vec{\mathfrak{O}}_a$	$\langle(0.9667, 0.8346); 0.2034, 3 \times 10^{-11}\rangle$
$\vec{\mathfrak{O}}_b$	$\langle(0.9453, 0.8608); 0.1587, 1 \times 10^{-14}\rangle$
$\vec{\mathfrak{O}}_c$	$\langle(0.8257, 0.6979); 0.3075, 3 \times 10^{-14}\rangle$
$\vec{\mathfrak{O}}_d$	$\langle(0.833, 0.7054); 0.2974, 5 \times 10^{-13}\rangle$
$\vec{\mathfrak{O}}_e$	$\langle(0.9082, 0.8501); 0.2478, 3 \times 10^{-14}\rangle$

Comparison for new and existing methods

This section will demonstrate the applicability and benefits of the suggested models by comparing them to existing models. The ED methodology was organized into four groups of methodologies. Distances can be grouped into the following categories: For example, the usage of fuzzy entropy or fuzzy knowledge measures may influence option ranking in a MADM situation. To demonstrate the utility and benefits of our models, we compare them to several current models. Our four weighted operators are based on ED and HD, respectively. A comparative analysis of the proposed models [26–28]. The novel ideas of interval-valued Pythagorean normal weighted averaging (IVPNWA), IVPNWG, GIVPNWA, and IVPNWG were presented by Zaoli et al [27]. Palanikumar et al. [26] deals that the ideas of GPyIVNNWA, GPyIVNNWG, PyIVNNWA and PyIVNNWG. Palanikumar et al. [28] introduced the idea of new types of WA, WG, GWA and GWG. As a result, the distances are as follows:

Table 6 Aggregated normalized decision values

	e_3	e_4
$\mathfrak{O}_a$	$\langle(0.8889, 0.8789); 0.1341, 0.892\rangle$	$\langle(0.8889, 0.7206); 0.1839, 0.2375\rangle$
$\mathfrak{O}_b$	$\langle(0.9444, 0.9412); 0.1174, 0.1341\rangle$	$\langle(0.9444, 0.7785); 0.1674, 0.1174\rangle$
$\mathfrak{O}_c$	$\langle(0.7778, 0.5402); 0.2862, 0.084\rangle$	$\langle(0.7778, 0.7101); 0.2169, 0.084\rangle$
$\mathfrak{O}_d$	$\langle(1, 0.7813); 0.1839, 0.3342\rangle$	$\langle(0.9444, 0.8858); 0.1674, 0.1174\rangle$
$\mathfrak{O}_e$	$\langle(0.8889, 0.7656); 0.1674, 0.1341\rangle$	$\langle(1, 0.9444); 0.2212, 0.1839\rangle$

Table 8 Different aggregating operator

Distance	WA	WG	GWA	GWG
$TOPSIS - HD$ (*proposed*)	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_b$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_b$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$
$TOPSIS - ED$ (*proposed*)	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_d \succ \mathfrak{O}_c$

Table 9 Ranking order of alternatives with different operators

Distance	IVPNWA	IVPNWG	GIVPNWA	GIVPNWG
$TOPSIS - HD$ [27]	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_c \succ \mathfrak{O}_d$	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_c$ $\mathfrak{O}_d \succ \mathfrak{O}_b$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_c \succ \mathfrak{O}_d$	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_c$ $\mathfrak{O}_d \succ \mathfrak{O}_b$
$TOPSIS - ED$ [27]	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_b \succ \mathfrak{O}_e$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_e \succ \mathfrak{O}_b$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_b \succ \mathfrak{O}_e$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_e \succ \mathfrak{O}_b$

During Eqs. (1) and (2) of the established method such as TPFNIWA, TPFNIWG, TGPFNIWA, and TGPFNIWG, the complete analysis by changing AOs is analyzed and their results are shown in Table 8.

Based on our observations of the aggregation activities, we provide more accurate, thorough, and creative outcomes for aggregating the proposed collection. Tables 9, 10, and 11 [26–28] provide the different AOs, such as WA, WG, GWA and GWG operators. The usefulness of these operators in situations when decision-makers want to modify their choice aggregation strategies to suit their own preferences is also examined in this work. The outcomes of using different operators are displayed in Tables 9, 10, and 11, which illustrates how decision-makers can make better choices by simultaneously considering expert opinions and assigned values. As demonstrated in the previous discussion, the suggested AOs provide decision makers with a more adaptable framework for choosing sound options. Additionally, these operators provide greater flexibility than traditional aggregation techniques. This demonstrates that the suggested operators offer greater flexibility and relevance in multiple contexts, while simultaneously managing a wider variety of DM scenarios. These AOs help decision-makers make well-informed choices that align with their requirements and preferences by offering a more flexible and inclusive framework. Additionally, these operators generalizability guarantees their use in a variety of decision domains, enhancing the DM process's overall robustness and reliability.

We contrast a number of current algorithms with the suggested approach to highlight its benefits. The ✓ and × symbols are used to indicate whether or not a property associated with the operator is satisfied. We may assess the validity and applicability of the proposed and current HD-based models by comparing them in Table 12.

Table 13 allows us to evaluate suggested and existing models based on EDs to determine their validity and applicability.

Because of these four interacting AOs, the HDs of the suggested and current methods are displayed in Fig. 2.

Because of these four interacting AOs, the EDs of the suggested and current methods are displayed in Fig. 3.

According to this comparison analysis, the suggested approach to DM issues is noticeably superior to those already in use.

Sensitivity analysis

The distance AO is used to determine the best option. The best outcomes were obtained by each technique with the highest value. The suggested aggregation algorithms superiority and validity were demonstrated by testing using the available tools. The suggested AO is more accurate and dependable than the current approach. As a new approach to choosing the optimal option, we proposed the AO for the MADM problem.

Table 10 Ranking order of alternatives with different operators

Distance	PyIVNSNWA	PyIVNNWG	GPyIVNNWA	GPyIVNNWG
$TOPSIS-HD$ [26]	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_c \succ \mathfrak{O}_d$	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_c$ $\mathfrak{O}_d \succ \mathfrak{O}_b$	$\mathfrak{O}_a \succ \mathfrak{O}_b \succ \mathfrak{O}_e$ $\mathfrak{O}_c \succ \mathfrak{O}_d$	$\mathfrak{O}_a \succ \mathfrak{O}_e \succ \mathfrak{O}_c$ $\mathfrak{O}_d \succ \mathfrak{O}_b$
$TOPSIS-ED$ [26]	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_b \succ \mathfrak{O}_e$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_e \succ \mathfrak{O}_b$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_b \succ \mathfrak{O}_e$	$\mathfrak{O}_a \succ \mathfrak{O}_c \succ \mathfrak{O}_d$ $\mathfrak{O}_e \succ \mathfrak{O}_b$

Table 11 Ranking order of alternatives with different operators

Distance	WA	WG	GWA	GWG
$TOPSIS-HD$ [28]	$\mathfrak{O}_e \succ \mathfrak{O}_d \succ \mathfrak{O}_c$ $\mathfrak{O}_b \succ \mathfrak{O}_a$	$\mathfrak{O}_e \succ \mathfrak{O}_c \succ \mathfrak{O}_b$ $\mathfrak{O}_a \succ \mathfrak{O}_d$	$\mathfrak{O}_e \succ \mathfrak{O}_d \succ \mathfrak{O}_c$ $\mathfrak{O}_b \succ \mathfrak{O}_a$	$\mathfrak{O}_e \succ \mathfrak{O}_c \succ \mathfrak{O}_b$ $\mathfrak{O}_a \succ \mathfrak{O}_d$
$TOPSIS-ED$ [28]	$\mathfrak{O}_e \succ \mathfrak{O}_d \succ \mathfrak{O}_c$ $\mathfrak{O}_b \succ \mathfrak{O}_a$	$\mathfrak{O}_e \succ \mathfrak{O}_c \succ \mathfrak{O}_b$ $\mathfrak{O}_a \succ \mathfrak{O}_d$	$\mathfrak{O}_e \succ \mathfrak{O}_d \succ \mathfrak{O}_c$ $\mathfrak{O}_b \succ \mathfrak{O}_a$	$\mathfrak{O}_e \succ \mathfrak{O}_c \succ \mathfrak{O}_b$ $\mathfrak{O}_a \succ \mathfrak{O}_d$

Table 12 Validity of the AO

HD	WA	WG	GWA	GWG
Yang et al. [27]	✓	✓	✓	✓
Palanikumar et al. [26]	✓	✓	✓	✓
Palanikumar et al. [28]	×	×	×	×

1. A novel method for PFS and PFNS that can handle uncertainty more precisely in practical settings is called an TPFNS. As a result, the suggested method is more appropriate than current methods for resolving issues in engineering and real life.
2. These operators are special instances of the aggregation operators that are currently in use. Our results show that compared to existing AOs, the ones provided here are more broad and more appropriate for real-world scenarios.

Advantages

The aforementioned study claims that the applications offer several benefits. We provide the sine trigonometric Pythagorean fuzzy number using NFN and sine trigonometric fuzzy numbers. An TPFNN may be used to understand humans and natural events, as well as to describe ambiguous data. TPFNN information is consequently more relevant to the human mind due to its broader reach. The different ranking results were achieved using the TPFNIWA, TPFNIWG, TGPFNIWA, and TGPFNIWG operators. Consequently, the results obtained using the novel TPFN interaction weighted aggregation operators are comparable to those obtained using the existing methods. As a result, it has been demonstrated that eliciting the information from a new type of TPFN interaction weighted aggregate information is more effective and successful. We use the aggregation operators of the TPFN interaction to determine the best choice. As a result, we can determine the best option for the implementation of decision support systems based on sine trigonometric operational principles by employing the recommended DM approach. To assess the suggested methods dependability and performance, we contrasted it with a few current algorithms. The domains or DM contexts in which the suggested method can be used may be restricted. A proposed strategy should be assessed in view of the constraints and circumstances that may maximize its effectiveness. The proposed technique includes certain assumptions and simplifications to facilitate analysis. The results may not always align with actual events, which would limit their usefulness or relevance. A proposed strategy should be assessed in view of the constraints and circumstances that may maximize its effectiveness. The proposed method makes several assumptions and simplifications to facilitate analysis. As a results, applicability would be limited because they might not always match actual events.

Conclusion

We investigated AOs in this work, with a particular emphasis on developing new sine trigonometric operation laws for Pythagorean normal fuzzy sets. Because of its symmetrical design and periodicity, the sine trigonometric function is particularly suitable for accommodating expert preferences throughout a variety of time periods. During DM procedures, clear operational laws are crucial. Our objective was to apply these characteristics to enhance DM and provide a more straightforward and meaningful option. For Pythagorean normal fuzzy numbers, we created sine trigonometric operation

Table 13 Validity of the AO

ED	WA	WG	GWA	GWG
Yang et al. [27]	✓	✓	✓	✓
Palanikumar et al. [26]	✓	✓	✓	✓
Palanikumar et al. [28]	×	×	×	×

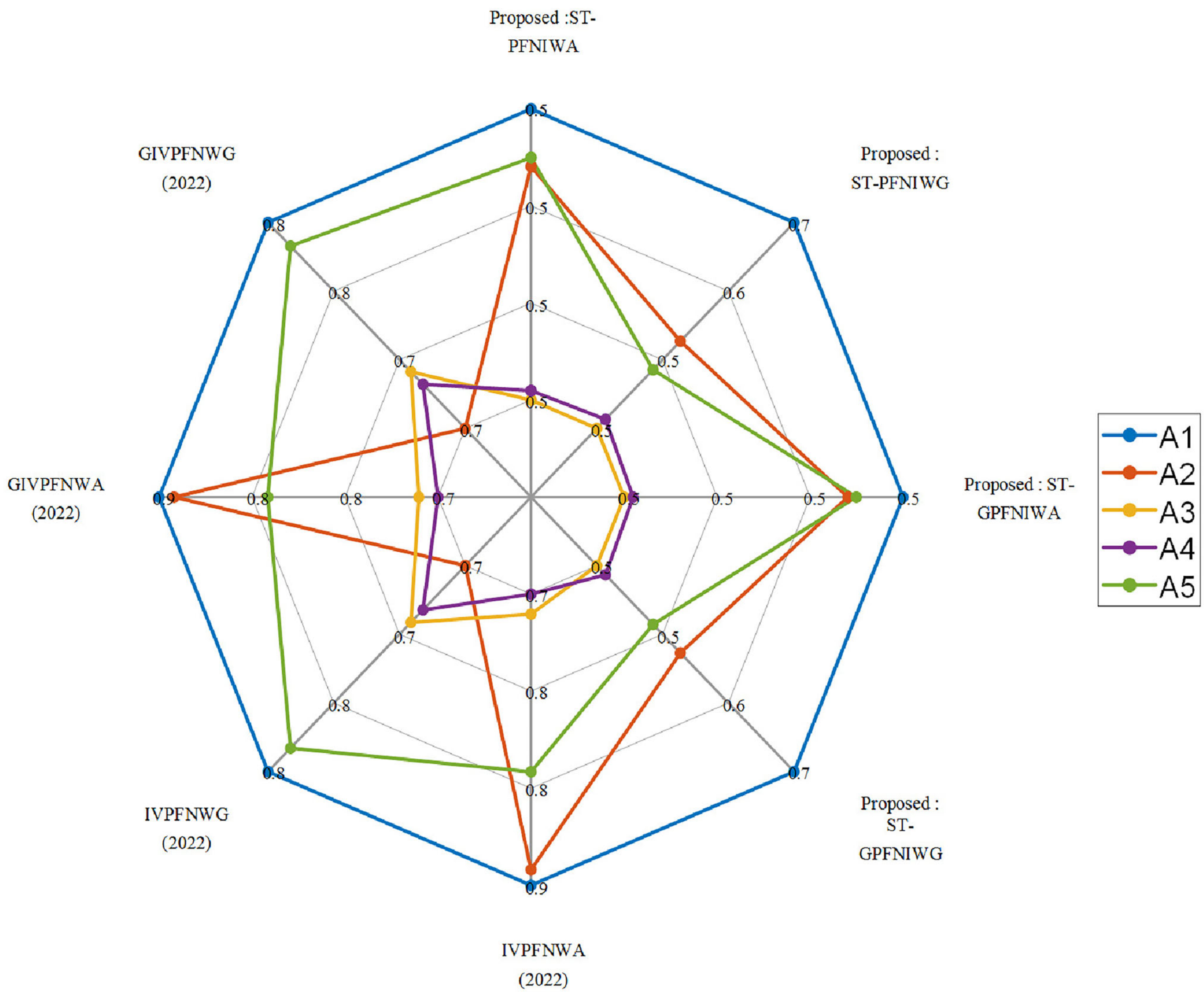

Fig. 2 HDs of proposed and existing approaches

laws and carefully examined their properties. Based on these ideas, we developed a number of average and geometric AOs that provide a framework for integrating the preferences of decision-makers. In order to provide a comprehensive picture of the AOs interactions, we also looked into their basic linkages. To apply these newly established ideas to real-world DM settings. Given that it considers the relationships between traits, it is clear that this method is effective. The suggested approach yields more precise rating outcomes. When considering the correlations between attributes, the proposed method is more efficient and better than [26–28] in handling real-world DM issues. A sine trigonometric FS with HD significantly improves data analysis. The advantages of HD are illustrated by a variety of compelling statistical examples.

This paper introduces new weighted operators, such as geometric and averaging operators. These operators possess a number of characteristics, including boundedness, idempotency, commutativity, associativity, and monotonicity.In order to characterize the weighted vector, we examined several traditional measures. Several AO criteria have been examined. The TPFNIWA, TPFNIWG, TGPFNIWA, and TGPFNIWG models and pictures were shown. The

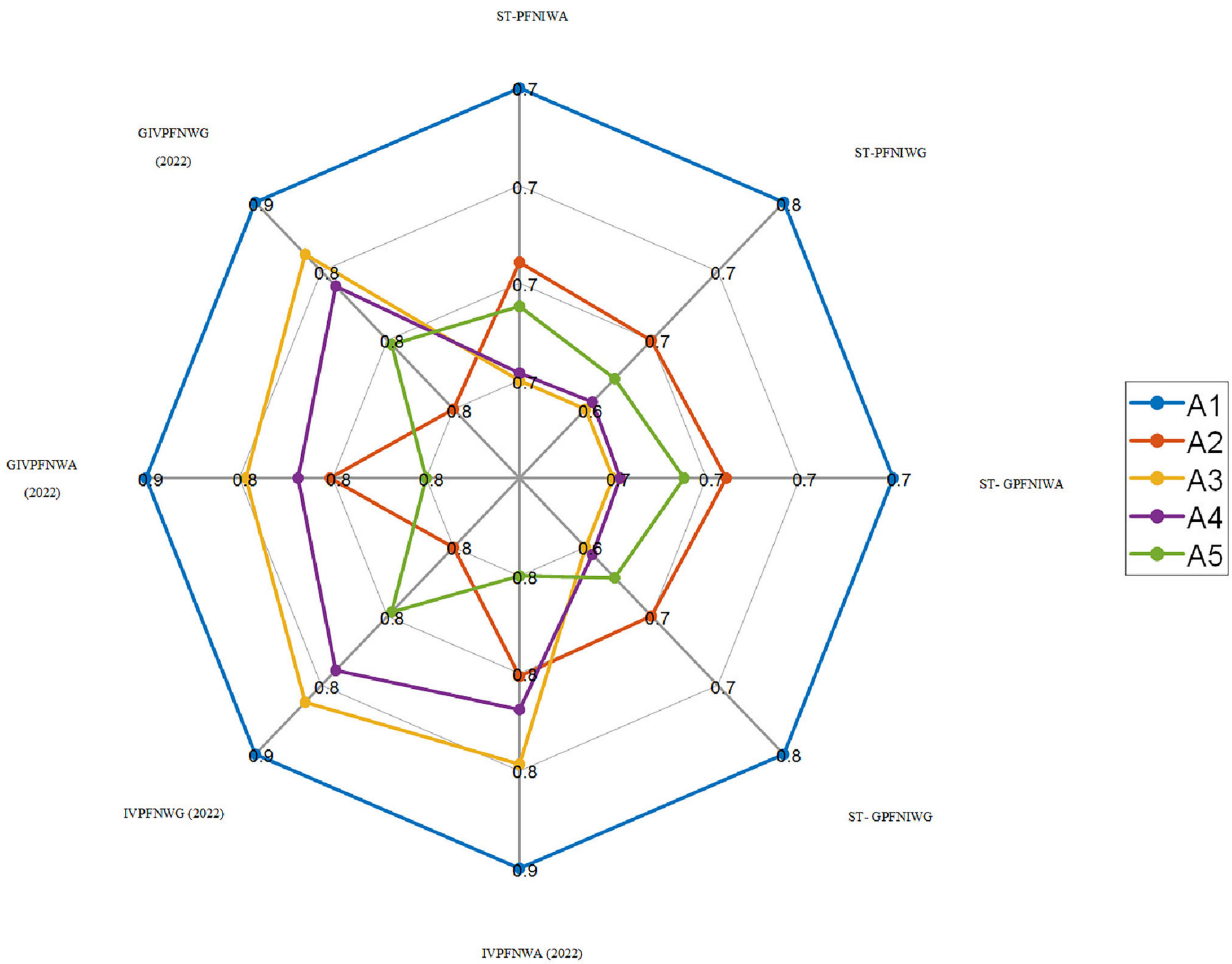

Fig. 3 EDs of proposed and existing approaches

TPFNN approach allows for the selection of a wide range of choices. Additionally, we use the TPFNIWA and TPFNIWG, TGPFNIWA and TGPFNIWG operators to discover different rankings. For DM problems a method for evaluating preferences with TPFNNs has been developed. The application of TPFNN information measures greatly enhanced a DM issue with uncertainty. The created method's usefulness, superiority, and practicality are tested on an industrial robot selection scenario. This work's performance was also contrasted with other others that already existed. Using a particular example of drive framework organization, we illustrated the efficacy and flexibility of the suggested approach. From choosing green suppliers to diagnosing medical conditions, this approach may be used to many different tasks. Future scholars will profit from the concepts discussed here because this topic is quite open. In the future, we will use the work based on new MADM models to address ambiguity and fuzziness in DM parameters. For solving MADM problems with given weights, we use interaction aggregating operators to study the similarity measures of q-rung FSs and (p, q)-rung FSs. There are several useful uses for HD and ED of NSs in data processing. If more research is done, these operators might be better than others like power mean AOs, Bonferroni mean operators, Heronian mean operators, etc. In further research, we might broaden our scope to encompass: (1) The sine trigonometric cubic FS and TIVPFS based on interaction AOs. (2) We will try to find the sine trigonometric normal vague set, sine trigonometric normal SFS and sine trigonometric normal NS. (3) The problem can be solved using complex TPFNIWA, complex TPFNIWG, complex TGPFNIWA and complex TGPFNIWG.

Appendix

Proof The proof of Theorem 3.1. (1) The proof follows from Definition 3.2. Here,

$$\Omega_1 \boxplus \Omega_2 = \begin{pmatrix} (\varrho_1+\varrho_2, \sigma_1+\sigma_2); \\ \sqrt{(\sin(\sphericalangle\cdot\zeta_1^t))^2+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\zeta_1^t))^2\cdot(\sin(\sphericalangle\cdot\zeta_2^t))^2}, \\ \sqrt{\begin{matrix}(\sin(\sphericalangle\cdot\eta_1^f))^2+(\sin(\sphericalangle\cdot\eta_2^f))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2\cdot(\sin(\sphericalangle\cdot\eta_2^f))^2 \\ -(\sin(\sphericalangle\cdot\eta_1^f))^2\cdot(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\zeta_1^t))^2\cdot(\sin(\sphericalangle\cdot\eta_2^f))^2\end{matrix}} \end{pmatrix}$$

$$= \begin{pmatrix} (\varrho_2+\varrho_1, \sigma_2+\sigma_1); \\ \sqrt{(\sin(\sphericalangle\cdot\zeta_2^t))^2+(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\zeta_2^t))^2\cdot(\sin(\sphericalangle\cdot\zeta_1^t))^2}, \\ \sqrt{\begin{matrix}(\sin(\sphericalangle\cdot\eta_2^f))^2+(\sin(\sphericalangle\cdot\eta_1^f))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2\cdot(\sin(\sphericalangle\cdot\eta_1^f))^2 \\ -(\sin(\sphericalangle\cdot\eta_2^f))^2\cdot(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\zeta_2^t))^2\cdot(\sin(\sphericalangle\cdot\eta_1^f))^2\end{matrix}} \end{pmatrix}$$

$$= \Omega_2 \boxplus \Omega_1.$$

(3) and,

$$\Omega_1 \boxtimes \Omega_2 = \begin{pmatrix} (\varrho_1\cdot\varrho_2, \sigma_1\cdot\sigma_2); \\ \sqrt{\begin{matrix}(\sin(\sphericalangle\cdot\zeta_1^t))^2+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\zeta_1^t))^2\cdot(\sin(\sphericalangle\cdot\zeta_2^t))^2 \\ -(\sin(\sphericalangle\cdot\zeta_1^t))^2\cdot(\sin(\sphericalangle\cdot\eta_2^f))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2\cdot(\sin(\sphericalangle\cdot\zeta_2^t))^2\end{matrix}}, \\ \sqrt{(\sin(\sphericalangle\cdot\eta_1^f))^2+(\sin(\sphericalangle\cdot\eta_2^f))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2\cdot(\sin(\sphericalangle\cdot\eta_2^f))^2} \end{pmatrix}$$

$$= \begin{pmatrix} (\varrho_2\cdot\varrho_1, \sigma_2\cdot\sigma_1); \\ \sqrt{\begin{matrix}(\sin(\sphericalangle\cdot\zeta_2^t))^2+(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\zeta_2^t))^2\cdot(\sin(\sphericalangle\cdot\zeta_1^t))^2 \\ -(\sin(\sphericalangle\cdot\zeta_2^t))^2\cdot(\sin(\sphericalangle\cdot\eta_1^f))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2\cdot(\sin(\sphericalangle\cdot\zeta_1^t))^2\end{matrix}}, \\ \sqrt{(\sin(\sphericalangle\cdot\eta_2^f))^2+(\sin(\sphericalangle\cdot\eta_1^f))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2\cdot(\sin(\sphericalangle\cdot\eta_1^f))^2} \end{pmatrix}$$

$$= \Omega_2 \boxtimes \Omega_1.$$

□

Proof The proof of Theorem 3.2. To prove that (3) as follows. By Eq. (1)

$$\left(\Delta_{\mathscr{E}}(\Omega_1,\Omega_2)+\Delta_{\mathscr{E}}(\Omega_2,\Omega_3)\right)^2$$

$$= \left[\begin{matrix} \sqrt{\begin{matrix}\left[\begin{matrix}\frac{1+(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2}{2}\varrho_1 \\ -\frac{1+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2}{2}\varrho_2\end{matrix}\right]^2 \\ +\frac{1}{2}\left[\begin{matrix}\frac{1+(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2}{2}\sigma_1 \\ -\frac{1+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2}{2}\sigma_2\end{matrix}\right]^2\end{matrix}} \\ +\sqrt{\begin{matrix}\left[\begin{matrix}\frac{1+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2}{2}\varrho_2 \\ -\frac{1+(\sin(\sphericalangle\cdot\zeta_3^t))^2-(\sin(\sphericalangle\cdot\eta_3^f))^2}{2}\varrho_3\end{matrix}\right]^2 \\ +\frac{1}{2}\left[\begin{matrix}\frac{1+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2}{2}\sigma_2 \\ -\frac{1+(\sin(\sphericalangle\cdot\zeta_3^t))^2-(\sin(\sphericalangle\cdot\eta_3^f))^2}{2}\sigma_3\end{matrix}\right]^2\end{matrix}} \end{matrix}\right]^2$$

implies

$$\begin{aligned}&\Big((\Xi_1\varrho_1-\Xi_2\varrho_2)^2+\frac{1}{2}(\Xi_1\sigma_1-\Xi_2\sigma_2)^2\Big)+\Big((\Xi_2\varrho_2-\Xi_3\varrho_3)^2+\frac{1}{2}(\Xi_2\sigma_2-\Xi_3\sigma_3)^2\Big)\\&\quad+2\Big(\sqrt{(\Xi_1\varrho_1-\Xi_2\varrho_2)^2+\frac{1}{2}(\Xi_1\sigma_1-\Xi_2\sigma_2)^2}\sqrt{(\Xi_2\varrho_2-\Xi_3\varrho_3)^2+\frac{1}{2}(\Xi_2\sigma_2-\Xi_3\sigma_3)^2}\ \Big)\end{aligned}$$

where

$$\begin{aligned}\Xi_1&=\frac{1+(\sin(\sphericalangle\cdot\zeta_1^t))^2-(\sin(\sphericalangle\cdot\eta_1^f))^2}{2},\\ \Xi_2&=\frac{1+(\sin(\sphericalangle\cdot\zeta_2^t))^2-(\sin(\sphericalangle\cdot\eta_2^f))^2}{2},\\ \Xi_3&=\frac{1+(\sin(\sphericalangle\cdot\zeta_3^t))^2-(\sin(\sphericalangle\cdot\eta_3^f))^2}{2}.\end{aligned}$$

Now, $\Big(\Delta_{\mathscr{E}}(\Omega_1,\Omega_2)+\Delta_{\mathscr{E}}(\Omega_2,\Omega_3)\Big)^2$

$$\begin{aligned}&\geq\Big((\Xi_1\varrho_1-\Xi_2\varrho_2)^2+\frac{1}{2}(\Xi_1\sigma_1-\Xi_2\sigma_2)^2\Big)+\Big((\Xi_2\varrho_2-\Xi_3\varrho_3)^2+\frac{1}{2}(\Xi_2\sigma_2-\Xi_3\sigma_3)^2\Big)\\&\quad+2\Big((\Xi_1\varrho_1-\Xi_2\varrho_2)\times(\Xi_2\varrho_2-\Xi_3\varrho_3)+\frac{1}{2}(\Xi_1\sigma_1-\Xi_2\sigma_2)\times(\Xi_2\sigma_2-\Xi_3\sigma_3)\Big)\\&=(\Xi_1\varrho_1-\Xi_2\varrho_2+\Xi_2\varrho_2-\Xi_3\varrho_3)^2+\frac{1}{2}(\Xi_1\sigma_1-\Xi_2\sigma_2+\Xi_2\sigma_2-\Xi_3\sigma_3)^2\\&=\Big[(\Xi_1\varrho_1-\Xi_3\varrho_3)^2+\frac{1}{2}(\Xi_1\sigma_1-\Xi_3\sigma_3)^2\Big]\\&=\Delta_{\mathscr{E}}(\Omega_1,\Omega_3)^2.\end{aligned}$$

□

Proof The proof of Theorem 4.1. The proof follows from mathematical induction.
By Eq. (3), If $n=2$, TPFNIWA$(\Omega_1,\Omega_2)=\xi_1\Omega_1\boxplus\xi_2\Omega_2$,
where,

$$\xi_1\Omega_1=\left(\begin{array}{c}(\xi_1\varrho_1,\xi_1\sigma_1);\sqrt{1-\Big(1-(\sin(\sphericalangle\cdot\zeta_1^t))^2\Big)^{\xi_1}},\\ \sqrt{\Big(1-(\sin(\sphericalangle\cdot\zeta_1^t))^2\Big)^{\xi_1}-\Big(1-(\sin(\sphericalangle\cdot\zeta_1^t)+\sin(\sphericalangle\cdot\eta_1^f))^2\Big)^{\xi_1}}\end{array}\right)$$

and

$$\xi_2\Omega_2=\left(\begin{array}{c}(\xi_2\varrho_2,\xi_2\sigma_2);\sqrt{1-\Big(1-(\sin(\sphericalangle\cdot\zeta_2^t))^2\Big)^{\xi_2}},\\ \sqrt{\Big(1-(\sin(\sphericalangle\cdot\zeta_2^t))^2\Big)^{\xi_2}-\Big(1-(\sin(\sphericalangle\cdot\zeta_2^t)+\sin(\sphericalangle\cdot\eta_2^f))^2\Big)^{\xi_2}}\end{array}\right)$$

We get $\xi_1\Omega_1 \boxplus \xi_2\Omega_2$

$$= \left(\begin{array}{c} (\xi_1\varrho_1 + \xi_2\varrho_2, \xi_1\sigma_1 + \xi_2\sigma_2); \\ \sqrt{\begin{array}{c} \left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_1^t))^2\right)^{\xi_1}\right) + \left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_2^t))^2\right)^{\xi_2}\right) \\ -\left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_1^t))^2\right)^{\xi_1}\right)\cdot\left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_2^t))^2\right)^{\xi_2}\right), \end{array}} \\ \sqrt{\begin{array}{r} \left(1-\left(1-(\sin(\sphericalangle\cdot\eta_1^f))^2\right)^{\xi_1}\right) + \left(1-\left(1-(\sin(\sphericalangle\cdot\eta_2^f))^2\right)^{\xi_2}\right) \\ -\left(1-\left(1-(\sin(\sphericalangle\cdot\eta_1^f))^2\right)^{\xi_1}\right)\cdot\left(1-\left(1-(\sin(\sphericalangle\cdot\eta_2^f))^2\right)^{\xi_2}\right) \\ -\left(1-(\sin(\sphericalangle\cdot\zeta_1^t)+\sin(\sphericalangle\cdot\eta_1^f))^2\right)^{\xi_1}\cdot\left(1-(\sin(\sphericalangle\cdot\zeta_2^t)+\sin(\sphericalangle\cdot\eta_2^f))^2\right)^{\xi_2} \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^2 \xi_j\varrho_j, \oslash_{j=1}^2\xi_j\sigma_j\right); \\ \sqrt{1-\bigcirc_{j=1}^2\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}}, \\ \sqrt{\bigcirc_{j=1}^2\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^2\left(1-(\sin(\sphericalangle\cdot\zeta_j^t)+\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}} \end{array} \right).$$

Using induction $n \geq 3$, TPFNIWA$(\Omega_1, \Omega_2, \ldots, \Omega_k)$

$$\left(\begin{array}{c} = \left(\oslash_{j=1}^k \xi_j\varrho_j, \oslash_{j=1}^k\xi_j\sigma_j\right); \\ \times\sqrt{1-\bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}}, \\ \times\sqrt{\bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t)+\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}}, \end{array} \right).$$

If $n = k+1$, $TPFNIWA(\Omega_1, \Omega_2, \ldots, \Omega_k, \Omega_{k+1})$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^k\xi_j\varrho_j + \xi_{k+1}\varrho_{k+1}, \oslash_{j=1}^k\xi_j\sigma_j + \xi_{k+1}\sigma_{k+1}\right); \\ \sqrt{\begin{array}{c} \oslash_{j=1}^k\left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}\right) + \left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_{k+1}^t))^2\right)^{\xi_{k+1}}\right) \\ -\bigcirc_{j=1}^k\left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}\right)\cdot\left(1-\left(1-(\sin(\sphericalangle\cdot\zeta_{k+1}^t))^2\right)^{\xi_{k+1}}\right), \end{array}} \\ \sqrt{\begin{array}{r} \oslash_{j=1}^k\left(1-\left(1-(\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}\right) + \left(1-\left(1-(\sin(\sphericalangle\cdot\eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \\ -\bigcirc_{j=1}^k\left(1-\left(1-(\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}\right)\cdot\left(1-\left(1-(\sin(\sphericalangle\cdot\eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \\ -\bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t)+\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}\cdot\left(1-(\sin(\sphericalangle\cdot\zeta_{k+1}^t)+\sin(\sphericalangle\cdot\eta_{k+1}^f))^2\right)^{\xi_{k+1}} \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^{k+1}\xi_j\varrho_j, \oslash_{j=1}^{k+1}\xi_j\sigma_j\right); \\ \sqrt{1-\bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}\cdot\left(1-(\sin(\sphericalangle\cdot\zeta_{k+1}^t))^2\right)^{\xi_{k+1}}}, \\ \sqrt{\begin{array}{r} \left[\bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^k\left(1-(\sin(\sphericalangle\cdot\zeta_j^t)+\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}\right]\cdot \\ \left[\left((\sin(\sphericalangle\cdot\zeta_{k+1}^t))^2\right)^{\xi_{k+1}} - (\sin(\sphericalangle\cdot\zeta_{k+1}^t)+\sin(\sphericalangle\cdot\eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right] \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^{k+1}\xi_j\varrho_j, \oslash_{j=1}^{k+1}\xi_j\sigma_j\right); \\ \sqrt{1-\bigcirc_{j=1}^{k+1}\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j}}, \\ \sqrt{\bigcirc_{j=1}^{k+1}\left(1-(\sin(\sphericalangle\cdot\zeta_j^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^{k+1}\left(1-(\sin(\sphericalangle\cdot\zeta_j^t)+\sin(\sphericalangle\cdot\eta_j^f))^2\right)^{\xi_j}} \end{array} \right)$$

□

Proof The proof of Theorem 4.2. Given that, $(\varrho_j, \sigma_j) = (\varrho, \sigma)$, $(\sin(\sphericalangle \cdot \zeta_j^t), \sin(\sphericalangle \cdot \eta_j^f)) = (\sin(\sphericalangle \cdot \zeta^t), \sin(\sphericalangle \cdot \eta^f))$ and $\oslash_{j=1}^n \xi_j = 1$. Then $TPFNIWA(\Omega_1, \Omega_2, \ldots, \Omega_n)$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^n \xi_j \varrho_j, \oslash_{j=1}^n \xi_j \sigma_j\right); \\ \sqrt{1 - \bigcirc_{j=1}^n \left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right)^{\xi_j}}, \\ \sqrt{\bigcirc_{j=1}^n \left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right)^{\xi_j} - \bigcirc_{j=1}^n \left(1 - (\sin(\sphericalangle \cdot \zeta^t) + \sin(\sphericalangle \cdot \eta^f))^2\right)^{\xi_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\varrho \oslash_{j=1}^n \xi_j, \sigma \oslash_{j=1}^n \xi_j\right); \\ \sqrt{1 - \left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right)^{\oslash_{j=1}^n \xi_j}}, \\ \sqrt{\left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right)^{\oslash_{j=1}^n \xi_j} - \left(1 - (\sin(\sphericalangle \cdot \zeta^t) + \sin(\sphericalangle \cdot \eta^f))^2\right)^{\oslash_{j=1}^n \xi_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} (\varrho, \sigma); \sqrt{1 - \left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right)}, \\ \sqrt{\left(1 - (\sin(\sphericalangle \cdot \zeta^t))^2\right) - \left(1 - (\sin(\sphericalangle \cdot \zeta^t) + \sin(\sphericalangle \cdot \eta^f))^2\right)} \end{array} \right)$$

$$= \left((\varrho, \sigma); \sin(\sphericalangle \cdot \zeta^t), \sin(\sphericalangle \cdot \eta^f)\right) = \Omega$$

□

Proof The proof of Theorem 4.3. Since, $\underline{\sin(\sphericalangle \cdot \zeta^t)} = \min \sin(\sphericalangle \cdot \zeta_{ji}^t)$, $\overline{\sin(\sphericalangle \cdot \zeta^t)} = \max \sin(\sphericalangle \cdot \zeta_{ji}^t)$ and $\underline{\sin(\sphericalangle \cdot \zeta^t)} \le \sin(\sphericalangle \cdot \zeta_{ji}^t) \le \overline{\sin(\sphericalangle \cdot \zeta^t)}$.

We have, $\underline{\sin(\sphericalangle \cdot \zeta^t)} = \sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\underline{\sin(\sphericalangle \cdot \zeta^t)})^2\right)^{\xi_j}} \le \sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\sin(\sphericalangle \cdot \zeta_{ji}^t))^2\right)^{\xi_j}}$
$\le \sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\overline{\sin(\sphericalangle \cdot \zeta^t)})^2\right)^{\xi_j}} = \overline{\sin(\sphericalangle \cdot \zeta^t)}$.

Since, $\underline{\sin(\sphericalangle \cdot \eta^f} = \min \sin(\sphericalangle \eta_{ji}^f$, $\overline{\sin(\sphericalangle \cdot \eta^f} = \max \sin(\sphericalangle \eta_{ji}^f$ and $\underline{\sin(\sphericalangle \cdot \eta^f} \le \sin(\sphericalangle \eta_{ji}^f \le \overline{\sin(\sphericalangle \cdot \eta^f}$. We have, $\underline{\sin(\sphericalangle \cdot \eta^f} = \bigcirc_{j=1}^n (\underline{\sin(\sphericalangle \cdot \eta^f})^{\xi_j} \le \bigcirc_{j=1}^n (\sin(\sphericalangle \eta_{ji}^f)^{\xi_j} \le \bigcirc_{j=1}^n (\overline{\sin(\sphericalangle \cdot \eta^f})^{\xi_j} = \overline{\sin(\sphericalangle \cdot \eta^f}$.

Since, $\underline{\varrho} = \min \varrho_{ji}, \overline{\varrho} = \max \varrho_{ji}, \underline{\sigma} = \max \sigma_{ji}, \overline{\sigma} = \min \sigma_{ji}$ and $\underline{\varrho} \le \varrho_{ji} \le \overline{\varrho}$ and $\overline{\sigma} \le \sigma_{ji} \le \underline{\sigma}$.

Hence, $\oslash_{j=1}^n \xi_j \underline{\varrho} \le \oslash_{j=1}^n \xi_j \varrho_{ji} \le \oslash_{j=1}^n \xi_j \overline{\varrho}$ and $\oslash_{j=1}^n \xi_j \overline{\sigma} \le \oslash_{j=1}^n \xi_j \sigma_{ji} \le \oslash_{j=1}^n \xi_j \underline{\sigma}$.

Therefore,

$$\oslash_{j=1}^n \xi_j \underline{\varrho} \times \left[\left(\sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\underline{\sin(\sphericalangle \cdot \zeta^t)})^2\right)^{\xi_j}} \right)^2 + 1 - \left(\bigcirc_{j=1}^n (\overline{\sin(\sphericalangle \cdot \eta^f)}^{\xi_j} \right)^2 \right]$$
$$\le \oslash_{j=1}^n \xi_j \varrho_{ji} \times \left[\left(\sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\sin(\sphericalangle \cdot \zeta_{ji}^t))^2\right)^{\xi_j}} \right)^2 + 1 - \left(\bigcirc_{j=1}^n (\sin(\sphericalangle \cdot \eta_{ji}^f)^{\xi_j} \right)^2 \right]$$
$$\le \oslash_{j=1}^n \xi_j \overline{\varrho} \times \left[\left(\sqrt[2]{1 - \bigcirc_{j=1}^n \left(1 - (\overline{\sin(\sphericalangle \cdot \zeta^t)})^2\right)^{\xi_j}} \right)^2 + 1 - \left(\bigcirc_{j=1}^n (\underline{\sin(\sphericalangle \cdot \eta^f)}^{\xi_j} \right)^2 \right].$$

Hence,

$$\left\langle (\underline{\varrho}, \underline{\sigma}); \underline{\sin(\sphericalangle \cdot \zeta^t)}, \overline{\sin(\sphericalangle \cdot \eta^f)} \right\rangle \le TPFNIWA(\Omega_1, \Omega_2, \ldots, \Omega_n)$$
$$\le \left\langle (\overline{\varrho}, \overline{\sigma}); \overline{\sin(\sphericalangle \cdot \zeta^t)}, \underline{\sin(\sphericalangle \cdot \eta^f)} \right\rangle.$$

□

Proof The proof of Theorem 4.4. For any j, $\varrho_{t_{ji}} \le \sigma_{h_{ji}}$. Therefore, $\oslash_{j=1}^n \varrho_{t_{ji}} \le \oslash_{j=1}^n \sigma_{h_{ji}}$.

For any j, $\left(\sin(\lessdot \cdot \zeta^t_{t_{ji}})\right)^2 \leq \left(\sin(\lessdot \cdot \zeta^t_{h_{ji}})\right)^2$.

Therefore, $1-\left(\sin(\lessdot \cdot \zeta^t_{t_j})\right)^2 \geq 1-\left(\sin(\lessdot \cdot \zeta^t_{h_j})\right)^2$.

Hence, $\bigcirc^n_{j=1}\left(1-\left(\sin(\lessdot \cdot \zeta^t_{t_j})\right)^2\right)^{\xi_j} \geq \bigcirc^n_{j=1}\left(1-\left(\sin(\lessdot \cdot \zeta^t_{h_j}\right)^2)\right)^{\xi_j}$ and $\sqrt[2]{1-\bigcirc^n_{j=1}\left(1-\left(\sin(\lessdot \cdot \zeta^t_{t_j})\right)^2\right)^{\xi_j}} \leq \sqrt[2]{1-\bigcirc^n_{j=1}\left(1-\left(\sin(\lessdot \cdot \zeta^t_{h_j})\right)^2\right)^{\xi_j}}$.

For any j, $\left(\sin(\lessdot \cdot \eta^f_{t_{ji}}\right)^2 \geq \left(\sin(\lessdot \cdot \eta^f_{h_{ji}}\right)^2$.

Therefore, $1-\left(\bigcirc^n_{j=1}\sin(\lessdot \cdot \eta^f_{t_{ji}}\right)^2 \leq 1-\left(\bigcirc^n_{j=1}\sin(\lessdot \cdot \eta^f_{h_{ji}}\right)^2$.

Hence,

$$\begin{aligned}
&\oslash^n_{j=1}\varrho_{tji} \times \left[\begin{array}{c}\left(\sqrt[2]{1-\bigcirc^n_{j=1}\left(1-(\sin(\lessdot \cdot \zeta^t_{tj}))^2\right)^{\xi_j}}\right)^2 \\ +1-\left(\bigcirc^n_{j=1}(\sin(\lessdot \cdot \eta^f_{tji})\right)^2\end{array}\right] \\
&\leq \oslash^n_{j=1}\varrho_{hji} \times \left[\left(\sqrt[2]{1-\bigcirc^n_{j=1}\left(1-(\sin(\lessdot \cdot \zeta^t_{hj}))^2\right)^{\xi_j}}\right)^2 + 1-\left(\bigcirc^n_{j=1}(\sin(\lessdot \cdot \eta^f_{hji})\right)^2\right].
\end{aligned}$$

Hence, $TPFNIWA\,(\Omega_1, \Omega_2, \ldots, \Omega_n) \leq TPFNIWA\,(W_1, W_2, \ldots, W_n)$. □

Proof The proof of Theorem 4.5. The proof is follows from mathematical induction.

By Eq. (4), If $n = 2$, $\mathrm{TPFNIWG}(\Omega_1, \Omega_2) = \Omega_1^{\xi_1} \boxtimes \Omega_2^{\xi_2}$, where,

$$\Omega_1^{\xi_1} = \left(\begin{array}{c}(\varrho_1^{\xi_1}, \sigma_1^{\xi_1}); \\ \sqrt{\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1} - \left(1-(\sin(\lessdot \cdot \zeta^t_1) + \sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}}, \\ \sqrt{1-\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}}\end{array}\right)$$

$$\Omega_2^{\xi_2} = \left(\begin{array}{c}(\varrho_2^{\xi_2}, \sigma_2^{\xi_2}); \\ \sqrt{\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2} - \left(1-(\sin(\lessdot \cdot \zeta^t_2) + \sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}}, \\ \sqrt{1-\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}}\end{array}\right)$$

We get,

$$\Omega_1^{\xi_1} \boxtimes \Omega_2^{\xi_2} = \left(\begin{array}{c}\left(\varrho_1^{\xi_1} \cdot \varrho_2^{\xi_2}, \sigma_1^{\xi_1} \cdot \sigma_2^{\xi_2}\right); \\ \sqrt{\begin{array}{r}\left(1-\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}\right) + \left(1-\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}\right) \\ -\left(1-\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}\right) \cdot \left(1-\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}\right) \\ -[\left(1-(\sin(\lessdot \cdot \zeta^t_1) + \sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1} \cdot \left(1-(\sin(\lessdot \cdot \zeta^t_2) + \sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}]\end{array}} \\ \sqrt{\begin{array}{r}\left(1-\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}\right) + \left(1-\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}\right) \\ -\left(1-\left(1-(\sin(\lessdot \cdot \eta^f_1))^2\right)^{\xi_1}\right) \cdot \left(1-\left(1-(\sin(\lessdot \cdot \eta^f_2))^2\right)^{\xi_2}\right)\end{array}}\end{array}\right)$$

Hence, $\mathrm{TPFNIWG}(\Omega_1, \Omega_2)$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{2} \varrho_j^{\xi_j}, \bigcirc_{j=1}^{2} \sigma_j^{\xi_j} \right); \\ \sqrt{\bigcirc_{j=1}^{2} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} - \bigcirc_{j=1}^{2} \left(1 - (\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}}, \\ \sqrt{1 - \bigcirc_{j=1}^{2} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}} \end{array} \right)$$

TPFNIWG$(\Omega_1, \Omega_2, \ldots, \Omega_k)$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k} \varrho_j^{\xi_j}, \bigcirc_{j=1}^{k} \sigma_j^{\xi_j} \right); \\ \sqrt{\bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} - \bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}}, \\ \sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}} \end{array} \right)$$

If $n = k + 1$, TPFNIWG$(\Omega_1, \ldots, \Omega_k, \Omega_{k+1})$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k} \varrho_j^{\xi_j} \cdot \varrho_{k+1}^{\xi_{k+1}}, \bigcirc_{j=1}^{k} \sigma_j^{\xi_j} \cdot \sigma_{k+1}^{\xi_{k+1}} \right); \\ \sqrt{\begin{array}{r} \oslash_{j=1}^{k} \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}\right) + \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \\ - \bigcirc_{j=1}^{k} \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}\right) \cdot \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \\ - \bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} \cdot \left(1 - (\sin(\lessdot \cdot \zeta_{k+1}^t) + \sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}} \end{array}} \\ \sqrt{\begin{array}{r} \oslash_{j=1}^{k} \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}\right) + \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \\ - \bigcirc_{j=1}^{k} \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}\right) \cdot \left(1 - \left(1 - (\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right) \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k+1} \varrho_j^{\xi_j}, \bigcirc_{j=1}^{k+1} \sigma_j^{\xi_j} \right); \\ \sqrt{\begin{array}{r} \left[\bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} - \bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}\right] \cdot \\ \left[\left((\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}} - (\sin(\lessdot \cdot \zeta_{k+1}^t) + \sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}\right] \end{array}}, \\ \sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} \cdot \left(1 - (\sin(\lessdot \cdot \eta_{k+1}^f))^2\right)^{\xi_{k+1}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k+1} \varrho_j^{\xi_j}, \bigcirc_{j=1}^{k+1} \sigma_j^{\xi_j} \right); \\ \sqrt{\bigcirc_{j=1}^{k+1} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j} - \bigcirc_{j=1}^{k+1} \left(1 - (\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}}, \\ \sqrt{1 - \bigcirc_{j=1}^{k+1} \left(1 - (\sin(\lessdot \cdot \eta_j^f))^2\right)^{\xi_j}} \end{array} \right)$$

$\square$

Proof The proof of Theorem 4.6. By Eq. (6), as a starting point, let as show that, $\oslash_{j=1}^{n} \xi_j \Omega_j^2 =$

$$\left(\begin{array}{c} \left(\left(\oslash_{j=1}^{n} \xi_j \varrho_j^2 \right), \left(\oslash_{j=1}^{n} \xi_j \sigma_j^2 \right) \right); \\ \sqrt{1 - \bigcirc_{j=1}^{n} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2\right)^2\right)^{\xi_j}}, \\ \sqrt{\bigcirc_{j=1}^{n} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2\right)^2\right)^{\xi_j} - \bigcirc_{j=1}^{n} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j}} \end{array} \right).$$

If $n = 2$, then $\xi_1 \Omega_1^2 =$

$$\left(\begin{array}{c}\left(\xi_1\varrho_1^2, \xi_1\sigma_1^2\right); \sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_1}}, \\ \sqrt{\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_1}-\left(1-\left((\sin(\lhd\cdot\zeta_1^t)+\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_1}}\end{array}\right) \text{ and}$$

$$\xi_2\Omega_2^2=\left(\begin{array}{c}\left(\xi_2\varrho_2^2, \xi_2\sigma_2^2\right); \sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_2^t))^2\right)^2\right)^{\xi_1}}, \\ \sqrt{\left(1-\left((\sin(\lhd\cdot\zeta_2^t))^2\right)^2\right)^{\xi_1}-\left(1-\left((\sin(\lhd\cdot\zeta_2^t)+\sin(\lhd\cdot\eta_2^f))^2\right)^2\right)^{\xi_1}}\end{array}\right).$$

We get, $\xi_1\Omega_1 \boxplus \xi_2\Omega_2$=

$$\left(\begin{array}{c}\left(\xi_1\varrho_1^2+\xi_2\varrho_2^2, \xi_1\sigma_1^2+\xi_2\sigma_2^2\right), \\ \sqrt{\begin{array}{c}\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_2^t))^2\right)^2\right)^{\xi_1}}\right)^2, \\ -\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\zeta_2^t))^2\right)^2\right)^{\xi_1}}\right)^2\end{array}} \\ \sqrt{\begin{array}{c}\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2+\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2, \\ -\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left[\begin{array}{c}\left(\sqrt{\begin{array}{r}\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_1}- \\ \left(1-\left((\sin(\lhd\cdot\zeta_1^t)+\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_1}\end{array}}\right)^2\cdot \\ \left(\sqrt{\begin{array}{r}\left(1-\left((\sin(\lhd\cdot\zeta_2^t))^2\right)^2\right)^{\xi_1}- \\ \left(1-\left((\sin(\lhd\cdot\zeta_2^t)+\sin(\lhd\cdot\eta_2^f))^2\right)^2\right)^{\xi_1}\end{array}}\right)^2\end{array}\right]\end{array}}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\oslash_{j=1}^2\,\xi_j\varrho_j^2, \oslash_{j=1}^2\xi_j\sigma_j^2\right), \\ \sqrt{1-\bigcirc_{j=1}^2\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_j}}, \\ \sqrt{\begin{array}{r}\bigcirc_{j=1}^2\left(1-\left((\sin(\lhd\cdot\zeta_1^t))^2\right)^2\right)^{\xi_j}- \\ \bigcirc_{j=1}^2\left(1-\left((\sin(\lhd\cdot\zeta_1^t)+\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_j}\end{array}}\end{array}\right)$$

In general,

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^{k} \xi_j \varrho_j^2, \oslash_{j=1}^{k} \xi_j \sigma_j^2 \right); \\ \sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t))^2 \right)^2 \right)^{\xi_j}} \\ \sqrt{\begin{array}{r} \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t))^2 \right)^2 \right)^{\xi_j} - \\ \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t) + \sin(\lessdot \cdot \eta_1^f))^2 \right)^2 \right)^{\xi_j} \end{array}} \end{array} \right)$$

If $n = k + 1$, $\oslash_{j=1}^{k} \xi_j \Omega_j^2 + \xi_{k+1} \Omega_{k+1}^2 = \oslash_{j=1}^{k+1} \xi_j \Omega_j^2$.
Now, $\oslash_{j=1}^{k} \xi_j \Omega_j^2 + \xi_{k+1} \Omega_{k+1}^2 = \xi_1 \Omega_1^2 \boxplus \xi_2 \Omega_2^2 \boxplus \ldots \boxplus \xi_k \Omega_k^2 \boxplus \xi_{k+1} \Omega_{k+1}^2$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^{k} \xi_j \varrho_j^2 + \xi_{k+1} \varrho_{k+1}^2, \oslash_{j=1}^{k} \xi_j \sigma_j^2 + \xi_{k+1} \sigma_{k+1}^2 \right); \\ \sqrt{\begin{array}{l} \left(\sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2 \right)^2 \right)^{\xi_j}} \right)^2 + \left(\sqrt{1 - \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t))^2 \right)^2 \right)^{\xi_1}} \right)^2, \\ - \left(\sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2 \right)^2 \right)^{\xi_j}} \right)^2 \cdot \left(\sqrt{1 - \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t))^2 \right)^2 \right)^{\xi_1}} \right)^2 \end{array}} \\ \sqrt{\begin{array}{l} \left(\sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2 \right)^2 \right)^{\xi_j}} \right)^2 + \left(\sqrt{1 - \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t))^2 \right)^2 \right)^{\xi_1}} \right)^2, \\ - \left(\sqrt{1 - \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2 \right)^2 \right)^{\xi_j}} \right)^2 \cdot \left(\sqrt{1 - \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t))^2 \right)^2 \right)^{\xi_1}} \right)^2 \\ - \left[\begin{array}{c} \left(\sqrt{\begin{array}{r} \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t))^2 \right)^2 \right)^{\xi_j} - \\ \bigcirc_{j=1}^{k} \left(1 - \left((\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2 \right)^2 \right)^{\xi_j} \end{array}} \right)^2 \cdot \\ \left(\sqrt{\begin{array}{r} \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t))^2 \right)^2 \right)^{\xi_1} - \\ \left(1 - \left((\sin(\lessdot \cdot \zeta_{k+1}^t) + \sin(\lessdot \cdot \eta_{k+1}^f))^2 \right)^2 \right)^{\xi_1} \end{array}} \right)^2 \end{array} \right] \end{array}} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\oslash_{j=1}^{k+1} \xi_j \varrho_j^2, \oslash_{j=1}^{k+1} \xi_j \sigma_j^2 \right); \\ \sqrt{1 - \bigcirc_{j=1}^{k+1} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t))^2 \right)^2 \right)^{\xi_j}}, \\ \sqrt{\begin{array}{r} \bigcirc_{j=1}^{k+1} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t))^2 \right)^2 \right)^{\xi_j} \\ - \bigcirc_{j=1}^{k+1} \left(1 - \left((\sin(\lessdot \cdot \zeta_1^t) + \sin(\lessdot \cdot \eta_1^f))^2 \right)^2 \right)^{\xi_j} \end{array}} \end{array} \right)$$

$$\text{and } \oslash_{j=1}^{k+1}\left(\xi_j \Omega_j^2\right)^{1/\Gamma} = \left(\begin{array}{c} \left(\left(\oslash_{j=1}^{k+1} \xi_j \varrho_j^2\right)^{1/\Gamma}, \left(\oslash_{j=1}^{k+1} \xi_j \sigma_j^2\right)^{1/\Gamma}\right); \\ \left(\sqrt{1 - \bigcirc_{j=1}^{k+1}\left(1 - \left((\sin(\sphericalangle \cdot \zeta_j^t))^2\right)^2\right)^{\xi_j}}\right)^{1/\Gamma}, \\ \left(\sqrt{\begin{array}{c} \bigcirc_{j=1}^{k+1}\left(1 - \left((\sin(\sphericalangle \cdot \zeta_j^t))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{k+1}\left(1 - \left((\sin(\sphericalangle \cdot \zeta_j^t) + \sin(\sphericalangle \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} \end{array}}\right)^{1/\Gamma} \end{array} \right). \quad \square$$

Proof The proof of Theorem 4.7. By Eq. (7), we prove this

$$\bigcirc_{j=1}^{n}(\Gamma\Omega_j)^{\xi_j} = \left(\begin{array}{c} \left(\bigcirc_{j=1}^{n} (\Gamma\varrho_j)^{\xi_j}, \bigcirc_{j=1}^{n}(\Gamma\sigma_j)^{\xi_j}\right); \\ \sqrt{\begin{array}{c} \bigcirc_{j=1}^{n}\left(1 - \left((\sin(\sphericalangle \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{n}\left(1 - \left((\sin(\sphericalangle \cdot \zeta_j^t) + \sin(\sphericalangle \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} \end{array}}, \\ \sqrt{1 - \bigcirc_{j=1}^{n}\left(1 - \left((\sin(\sphericalangle \cdot \eta_j^f))^2\right)^2\right)^{\xi_j}} \end{array} \right)$$

If $n = 2$, then

$$(\Gamma\Omega_1)^{\xi_1} = \left(\begin{array}{c} \left((\Gamma\varrho_1)^{\xi_1}, (\Gamma\sigma_1)^{\xi_1}\right); \\ \sqrt{\left(1 - \left((\sin(\sphericalangle \cdot \eta_1^f))^2\right)^2\right)^{\xi_1} - \left(1 - \left((\sin(\sphericalangle \cdot \zeta_1^t) + \sin(\sphericalangle \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}} \\ \sqrt{1 - \left(1 - \left((\sin(\sphericalangle \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}} \end{array} \right)$$

$$\text{and } (\Gamma\Omega_2)^{\xi_2} = \left(\begin{array}{c} \left((\Gamma\varrho_2)^{\xi_2}, (\Gamma\sigma_2)^{\xi_2}\right); \\ \sqrt{\left(1 - \left((\sin(\sphericalangle \cdot \eta_2^f))^2\right)^2\right)^{\xi_1} - \left(1 - \left((\sin(\sphericalangle \cdot \zeta_2^t) + \sin(\sphericalangle \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}} \\ \sqrt{1 - \left(1 - \left((\sin(\sphericalangle \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}} \end{array} \right)$$

We get, $(\Gamma\Omega_1)^{\xi_1} \boxtimes (\Gamma\Omega_2)^{\xi_2}$

$$= \left(\begin{array}{c} \left((\Gamma\varrho_1)^{\xi_1} \cdot (\Gamma\varrho_2)^{\xi_2}, (\Gamma\sigma_1)^{\xi_1} \cdot (\Gamma\sigma_2)^{\xi_2} \right); \\ \sqrt{ \begin{array}{c} \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2 + \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2 \cdot \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ - \left[\begin{array}{c} \left(\sqrt{ \begin{array}{r} \left(1-\left((\sin(\lessdot \cdot \eta_1^f))^2\right)^2\right)^{\xi_1} - \\ \left(1-\left((\sin(\lessdot \cdot \zeta_1^t) + \sin(\lessdot \cdot \zeta_1^t))^2\right)^2\right)^{\xi_1} \end{array} } \right)^2 \cdot \\ \left(\sqrt{ \begin{array}{r} \left(1-\left((\sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1} - \\ \left(1-\left((\sin(\lessdot \cdot \zeta_2^t) + \sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1} \end{array} } \right)^2 \end{array} \right], \end{array} } \\ \sqrt{ \begin{array}{c} \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2 + \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_1^f))^2\right)^2\right)^{\xi_1}}\right)^2 \cdot \left(\sqrt{1-\left(1-\left((\sin(\lessdot \cdot \eta_2^f))^2\right)^2\right)^{\xi_1}}\right)^2 \end{array} } \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{2} (\Gamma\varrho_j)^{\xi_j}, \bigcirc_{j=1}^{2} (\Gamma\sigma_j)^{\xi_j} \right); \\ \sqrt{ \begin{array}{r} \bigcirc_{j=1}^{2} \left(1-\left((\sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{2} \left(1-\left((\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} \end{array} }, \\ \sqrt{1-\bigcirc_{j=1}^{2} \left(1-\left((\sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j}} \end{array} \right)$$

If $n = k$, $\bigcirc_{j=1}^{k} (\Gamma\varrho_j)^{\xi_j} =$

$$\left(\begin{array}{c} \left(\bigcirc_{j=1}^{k} (\Gamma\varrho_j)^{\xi_j}, \bigcirc_{j=1}^{k} (\Gamma\sigma_j)^{\xi_j} \right); \\ \sqrt{ \begin{array}{r} \bigcirc_{j=1}^{k} \left(1-\left((\sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{k} \left(1-\left((\sin(\lessdot \cdot \zeta_j^t) + \sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j} \end{array} }, \\ \sqrt{1-\bigcirc_{j=1}^{k} \left(1-\left((\sin(\lessdot \cdot \eta_j^f))^2\right)^2\right)^{\xi_j}} \end{array} \right)$$

If $n = k+1$, then $\bigcirc_{j=1}^{k}(\Gamma\Omega_j)^{\xi_j}\cdot(\Gamma\Omega_{k+1})^{\xi_{k+1}} = \bigcirc_{j=1}^{k+1}(\Gamma\Omega_j)^{\xi_j}$.
Now, $\bigcirc_{j=1}^{k}(\Gamma\Omega_j)^{\xi_j}\cdot(\Gamma\Omega_{k+1})^{\xi_{k+1}} = (\Gamma\Omega_1)^{\xi_1}\boxtimes(\Gamma\Omega_2)^{\xi_2}\boxtimes\ldots\boxtimes(\Gamma\Omega_k)^{\xi_k}\boxtimes(\Gamma\Omega_{k+1})^{\xi_{k+1}}$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k}(\Gamma\varrho_j)^{\xi_j}\cdot(\Gamma\varrho_{k+1})^{\xi_{k+1}}, \bigcirc_{j=1}^{k}(\Gamma\sigma_j)^{\xi_j}\cdot(\Gamma\sigma_{k+1})^{\xi_{k+1}}\right); \\ \dfrac{\sqrt{\begin{array}{c} \left(\sqrt{1-\bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^2 + \left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left(\sqrt{1-\bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left[\begin{array}{c} \left(\sqrt{\begin{array}{r} \bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\zeta_j^t)+\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j} \end{array}}\right)^2\cdot \\ \left(\sqrt{\begin{array}{r} \left(1-\left((\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1} - \\ \left(1-\left((\sin(\lhd\cdot\zeta_{k+1}^t)+\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1} \end{array}}\right)^2 \end{array}\right] \end{array}}}{\sqrt{\begin{array}{c} \left(\sqrt{1-\bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^2 + \left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1}}\right)^2 \\ -\left(\sqrt{1-\bigcirc_{j=1}^{k}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^2\cdot\left(\sqrt{1-\left(1-\left((\sin(\lhd\cdot\eta_{k+1}^f))^2\right)^2\right)^{\xi_1}}\right)^2 \end{array}}}, \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k+1}(\Gamma\varrho_j)^{\xi_j}, \bigcirc_{j=1}^{k+1}(\Gamma\sigma_j)^{\xi_j}\right); \\ \dfrac{\sqrt{\begin{array}{r} \bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\zeta_1^t)+\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_j} \end{array}}}{\sqrt{1-\bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\eta_1^f))^2\right)^2\right)^{\xi_j}}}, \end{array}\right)$$

and $\frac{1}{\Gamma}\left(\bigcirc_{j=1}^{k+1}(\Gamma\Omega_j)^{\xi_j}\right)$

$$= \left(\begin{array}{c} \left(\bigcirc_{j=1}^{k+1}(\Gamma\varrho_j)^{\xi_j}, \bigcirc_{j=1}^{k+1}(\Gamma\sigma_j)^{\xi_j}\right); \\ \dfrac{\left(\sqrt{\begin{array}{r} \bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j} - \\ \bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\zeta_j^t)+\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j} \end{array}}\right)^{1/\Gamma}}{\left(\sqrt{1-\bigcirc_{j=1}^{k+1}\left(1-\left((\sin(\lhd\cdot\eta_j^f))^2\right)^2\right)^{\xi_j}}\right)^{1/\Gamma}}, \end{array}\right)$$

□

Author Contributions Murugan Palanikumar: conceptualization, methodology, software, validation, formal analysis, investigation, data curation, visualization, writing—original draft. Nasreen Kausar: methodology, software, validation, formal analysis, investigation, writing—original draft. Dragan Pamucar: writing-review and editing. Vladimir Simic: writing-review and editing.

Data availability The data used to support the findings of this study are included within the article.

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