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NOTE ON M^* -GROUPS¹

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Abstract

A compact Klein surface with boundary of algebraic genus $g \geq 2$ has at most $12(g - 1)$ automorphisms. When a surface attains this bound, it has maximal symmetry, and the group of automorphisms is then called an M^* -group. In this paper, we consider the extended modular group $\bar{\Gamma}$ and we give some new examples of M^* -groups which are finite quotient groups of $\bar{\Gamma}$.

1. Introduction

In [8], May proved that if X is a compact bordered Klein Surface of algebraic genus $g \geq 2$, then the automorphism group G of X is finite, and the order of G is at most $12(g - 1)$. If the order of G is the largest possible, then G is called an M^* -group. Thus a finite group G is called an M^* -group if it is generated by three distinct non-trivial elements α, β, γ which satisfy

$$\alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^2 = (\alpha\gamma)^3 = I$$

and other relations which make the group finite. May proved that there are M^* -groups for infinitely many g [9], and that there are no M^* -groups for infinitely many g [10]. Additional results about these groups are given in [2, 3, 5, 6, 11, 12, 13, 14, 17]. The reference [1] contains a nice survey of known results about M^* -groups.

On the other hand, we briefly mention about the extended modular group. The modular group Γ is the discrete subgroup of $PSL(2, \mathbb{R})$

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generated by two linear fractional transformations

$$t(z) = -\frac{1}{z} \quad \text{and} \quad s(z) = -\frac{1}{z+1}.$$

Then modular group Γ has a presentation

$$\Gamma = \langle t, s \mid t^2 = s^3 = I \rangle \cong C_2 * C_3.$$

By adding the reflection $r(z) = 1/\bar{z}$ to the generators of the modular group Γ , the extended modular group $\bar{\Gamma}$ has been defined in [4]. The extended modular group $\bar{\Gamma}$ has a presentation

$$\bar{\Gamma} = \langle t, s, r \mid t^2 = s^3 = r^2 = I, rt = tr, rs = s^{-1}r \rangle$$

or, equivalently,

$$\bar{\Gamma} = \langle t, s, r \mid t^2 = s^3 = r^2 = (rt)^2 = (rs)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_3.$$

If we take

$$t = r_2 r_1 = r_1 r_2, \quad s = r_1 r_3, \quad \text{and} \quad r = r_1,$$

where

$$r_1(z) = \frac{1}{\bar{z}}, \quad r_2(z) = -\bar{z}, \quad \text{and} \quad r_3(z) = \frac{-\bar{z}}{\bar{z}+1},$$

then, we have

$$\langle r_1, r_2, r_3 \mid r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^2 = (r_1 r_3)^3 = I \rangle.$$

Here r_1, r_2 and r_3 have matrix representations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively (in this work, we identify each matrix A in $GL(2, \mathbb{Z})$ with $-A$, so that they each represent the same element of $PGL(2, \mathbb{Z})$). Thus the modular group $\Gamma = PSL(2, \mathbb{Z})$ is a subgroup of index 2 in the extended modular group $\bar{\Gamma}$.

Jones and Thornton defined in [7] that the function

$$\alpha : r_1 \rightarrow r_1, \quad r_2 \rightarrow r_1 r_2, \quad r_3 \rightarrow r_3$$

or

$$\alpha : t \rightarrow rt, \quad s \rightarrow s, \quad r \rightarrow r$$

is an automorphism of $\bar{\Gamma}$ which is not inner. Then the outer automorphism class group $Out\bar{\Gamma} = Aut\bar{\Gamma}/Inn\bar{\Gamma}$ is generated by α and its order is 2. Thus, with only finitely many exceptions, α maps congruence subgroups, to non-congruence subgroups of $\bar{\Gamma}$.

The role of the extended modular group $\bar{\Gamma}$ in the study of M^* -groups follows from the following lemmas.

Lemma 1.1 ([9]). *A finite group G of order at least 12 is an M^* -group if and only if G is the homomorphic image of the extended modular group $\bar{\Gamma}$.*

A trivial consequent of this lemma, if the order of quotient group of $\bar{\Gamma}$ is greater than 6, then the homomorphic image of it is an M^* -group.

When we were studying the power subgroups of the extended modular group $\bar{\Gamma}$ in [16], we became aware of the fact that the homomorphic images of the quotient groups of power subgroups of the extended modular group $\bar{\Gamma}$ are M^* -groups.

In this paper, we give some new examples of M^* -groups which are the homomorphic images of the quotient groups of some normal subgroups of the extended modular group $\bar{\Gamma}$.

2. The Results

The study of M^* -groups lies in the study of factor groups of the extended modular group $\bar{\Gamma}$. We will need later the following technical lemmas describing certain subgroups of $\bar{\Gamma}$.

Lemma 2.1. *There are exactly 2 normal subgroups of index 12 in $\bar{\Gamma}$. Explicitly these are $\Gamma' = \langle tsts^2, ts^2ts \rangle$ and $\Gamma_2 = \langle tsts, ts^2ts^2 \rangle$.*

Proof. It is known [15], that the commutator subgroup Γ' is a normal subgroup of index 6 in Γ and the principal congruence subgroup Γ_2 is a normal subgroup of index 6 in Γ . Since $|\bar{\Gamma} : \Gamma| = 2$, we have $|\bar{\Gamma} : \Gamma'| = |\bar{\Gamma} : \Gamma| |\Gamma : \Gamma'| = 2 \cdot 6 = 12$ and $|\bar{\Gamma} : \Gamma_2| = |\bar{\Gamma} : \Gamma| |\Gamma : \Gamma_2| = 2 \cdot 6 = 12$.

Also we find $\Gamma' = \langle tsts^2, ts^2ts \rangle$ and $\Gamma_2 = \langle tsts, ts^2ts^2 \rangle$. □

Lemma 2.2 ([16]). *(i) The subgroup $\bar{\Gamma}^6$ is a normal subgroup of $\bar{\Gamma}$ of index 432.*

(ii) For $1 \leq m \leq 72$, the subgroup $\bar{\Gamma}^{6m}$ is a normal subgroup of finite index in $\bar{\Gamma}$ and $\bar{\Gamma}^{6m} = \Gamma^{6m}$ where $\bar{\Gamma}^m$ is called the m -th power subgroup – the subgroup of $\bar{\Gamma}$ generated by the m -th powers of all elements of $\bar{\Gamma}$.

Thus we can give the following results:

Corollary 2.3. *(i) The quotient groups $\bar{\Gamma}/\Gamma'$ and $\bar{\Gamma}/\Gamma_2$ are M^* -groups.*

(ii) The quotient group $\bar{\Gamma}/\Gamma^6$ is an M^ -group.*

Theorem 2.4. *For each integer $1 \leq m \leq 72$, the subgroup $\bar{\Gamma}/\Gamma^{6m}$ is an M^* -group.*

Now for obtaining new infinitive family of M^* -groups, we use the results of Jones and Thornton on automorphisms and congruence subgroups of the extended modular group $\bar{\Gamma}$ [7].

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent a typical element of $\bar{\Gamma}$. For each integer $n \geq 1$, we define

$$\begin{aligned}\bar{\Gamma}_n &= \{A \in \bar{\Gamma} \mid a \equiv d \equiv \pm 1 \text{ and } b \equiv c \equiv 0 \pmod{n}\}, \\ M_n &= \{A \in \bar{\Gamma} \mid a \equiv d \text{ and } b \equiv c \equiv 0 \pmod{n}\}, \\ \Gamma_n &= \bar{\Gamma}_n \cap \Gamma, \\ N_n &= M_n \cap \Gamma.\end{aligned}$$

These are normal subgroups of finite index in $\bar{\Gamma}$, and called the *principal congruence subgroups*. Also, Jones and Thornton proved that the following results :

If $n \geq 2$ then $M_n \geq N_n \geq \Gamma_n$ and $M_n \geq \bar{\Gamma}_n \geq \Gamma_n$. Also if $n > 2$ then $\bar{\Gamma}_n = \Gamma_n$ and if $n = 2$ then $\bar{\Gamma}_2 \geq \Gamma_2 \geq \bar{\Gamma}_4 = \Gamma_4$. Thus a subgroup K of $\bar{\Gamma}$ contains some $\bar{\Gamma}_n$ if and only if it contains some Γ_n ; such a subgroup K is called a *congruence subgroup* and the level of K is the least n such that $K \geq \Gamma_n$.

If n is divisible by 4 or by a prime $p \equiv 3 \pmod{4}$ then $M_n = N_n$ and also if n is even then $N_n \leq N_2 = \Gamma_2$. Additionally to these, if n is 1, 2, 4, 6, or an odd prime-power, $N_n = \bar{\Gamma}_n$.

There are 8 normal subgroups of $\bar{\Gamma}$ containing N_{12} such that their indexes are greater than 12. These are: two normal subgroups of index 24 in $\bar{\Gamma}$ with representatives $N_3 = M_3$ and $N_3\alpha$; one normal subgroup of index 36 in $\bar{\Gamma}$ with representative $\bar{\Gamma}''$; two normal subgroups of index 48 in $\bar{\Gamma}$ with representatives $N_4 = M_4$ and N_3^* which is a normal subgroup of index 2 in N_3 ; two normal subgroups of index 144 in $\bar{\Gamma}$ with representatives N_6 and $N_6\alpha$; one normal subgroup of index 576 in $\bar{\Gamma}$ with representative N_{12} .

On the other hand in [9], May proved the following theorem :

Theorem 2.5 ([9]). *For each integer $n \geq 2$, $\bar{\Gamma}/\Gamma_n$ is an M^* -group.*

Using the above results of Jones and Thornton, we can generalize this theorem of May to all congruence subgroups of $\bar{\Gamma}$.

Theorem 2.6. (i) *For each integer $n > 2$, $\bar{\Gamma}/\bar{\Gamma}_n$ is an M^* -group.*

(ii) *For each even integer $n \geq 2$, the subgroup $\bar{\Gamma}/N_n$ is an M^* -group.*

(iii) *If n is divisible by 4 or by a prime $p \equiv 3 \pmod{4}$ then $\bar{\Gamma}/M_n$ is an M^* -group.*

(iv) *If n is 1, 2, 4, 6, or an odd prime-power, $\bar{\Gamma}/N_n$ is an M^* -group.*

Corollary 2.7. *The subgroups $\bar{\Gamma}/N_3$, $\bar{\Gamma}/N_3\alpha$, $\bar{\Gamma}/\bar{\Gamma}''$, $\bar{\Gamma}/N_4$, $\bar{\Gamma}/N_3^*$, $\bar{\Gamma}/N_6$, $\bar{\Gamma}/N_6\alpha$, and $\bar{\Gamma}/N_{12}$ are M^* -groups.*

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